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ISSUE No. 13

PRACTICAL MATHEMATICS

THEORY AND PRACTICE WITH MILITARY AND INDUSTRIAL APPLICATIONS

APPLIED MATHEMATICS

Navigation

Position and Distance
Piloting • Direction Finding
Great-Circle Sailing
Aerial Navigation

Radio

Television and
High-Frequency Transmission
Special Circuits

—ALSO—

Mathematical Tables and Formulas
Self-Tests and Problems

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B.S. in C.E.
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35¢

EDITOR: REGINALD STEVENS KIMBALL ED.D.

Issue No. 13 PRACTICAL MATHEMATICS Volume No. 2

ISSUE 13 **Practical Mathematics** VOLUME 2

REGINALD STEVENS KIMBALL, Editor

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CHATS WITH THE EDITOR

AS WE go into the last stretch, with this thirteenth issue of PRACTICAL MATHEMATICS, we take up the last two of the applied fields—navigation and radio—as projected for our course. We follow here the same practice that we have used in the other issues devoted to applied mathematics, treating the computational aspects of the subject without going into complete details regarding the development of the subjects themselves.

The National Educational Alliance member who studies these issues carefully should come away from his study with some very convenient "rule of thumb" suggestions which he can put to immediate use in his own field of activity. It is not to be expected that he will know all there is to know about the subject, as each one of these subjects would require several volumes to give the complete treatment of theory.

Navigation plays an increasingly large part in human affairs. The development of the airplane has extended the need for navigation and a complete new science—avigation, or air navigation—has grown up within the present century. Since the general principles of air navigation are much the same as those of sea navigation, we may safely begin our study with the older forms of navigational practice, merely noting some of the important points at which air navigation differs. In his treatment of navigation, Dr. Michael finds it necessary to step aside and provide some definitions from the fields of astronomy and spherical trigonometry, in order to give his readers a complete background for understanding the manner in which the formulas are developed.

With the rapidity of motion possible in the modern airplane, the pilot would be at a loss if he had to stop to go through lengthy computations every time he wanted to determine his position. Fortunately, tables, charts, and calculating devices have been perfected which eliminate most of the "figuring" which is involved, both for the pilot in the air and for the pilot on the sea. If you are actively engaged in either of these fields of transportation, or if you expect some time to be so engaged, you will want to familiarize yourself with the available devices. Meantime, be assured that you will be in a better position to appreciate their use and to put them to good use if you first follow through the computations which they, to some extent, eliminate.

No field has grown more rapidly in the period since the First World War than the radio industry. At the present time, our armed forces are giving training in radio to 15 out of every 1000 men in uniform. The demands for a knowledge of radio in war-time are rather great; if we may judge from the experience at the close of World War I, the demands in the peace which is to follow the war will be even greater. Already, manufacturers are champing at the bit, desirous of getting under way with improvements in the present receiving and sending sets, eager to put television on a basis for popular consumption, ready to print your morning newspaper in your own home

while you are yet asleep, and carrying on explorations into many other phases of radio development of which the general public is as yet uninformed.

Mr. Maedel is in the fortunate position of being connected with one of the important corporations dealing with radio. In his daily classroom contacts with young men who are studying to enter the industry, he is giving just the kind of practical training which will enable them to glean from the whole field of mathematics those essential parts of the subject for which they will have need. He summarizes this training briefly in his article in this issue.

As promised you in the first issue, we are devoting Issue Number Fourteen to general applications of mathematics. In that issue, which will reach you in about ten days, we are going to provide a general review of the course thus far. We have proceeded through this course in systematic fashion, following the various fields of mathematics separately and noting how each one depends upon those which have gone before for its development. Having progressed through the calculus and differential equations, we next took a look at eight of the important fields in which a knowledge of mathematics is of paramount importance. In the space of half an issue apiece, we have discussed the mathematics of heat, the mathematics of chemistry, the mathematics of construction engineering, the mathematics of machine-shop practice, the mathematics of electricity, the mathematics of gunnery, the mathematics of navigation, and the mathematics of radio. As a capstone to the course, in the review issue which is yet to come, we shall select a project in each of these fields and point out exactly how one would draw upon his knowledge of mathematics to solve a specific problem. Through the completeness of the illustrative examples chosen for this purpose, we expect to throw further light on difficulties which you may have encountered in trying to solve some of the practice exercises in the various issues which have gone before.

The fourteenth issue will also contain a comprehensive index to all that has preceded, so that you may readily turn to the exact page on which you may find the discussion of any point upon which you wish to refresh your memory.

When we were laying out the work in PRACTICAL MATHEMATICS, we were prompted by the many requests which the National Educational Alliance had been receiving to provide at nominal expense material which would assist men and women to prepare themselves for entrance into the armed services or into some field of industry connected with the war effort. As indicated in the subtitle of the magazine, "Theory and Practice, with Military and Industrial Applications", we purposely confined our attention to just those phases of mathematics which were essential to the furtherance of the immediate aims of winning the war.

Now it is time to look toward the days of peace, which we may hope are not too far distant. The problems of the post-war period will demand men and women trained in the mathematics of business. Already, we have begun to receive requests from members of the Alliance who are far-sightedly looking toward the time when they may be called upon to assist in the development of the business organization of the industry with which they are connected. Men now in uniform, upon their return to civilian life, will find also that a knowledge of business mathematics will be of great help to them in furthering their selection for key positions.

Those of you who have been following this series of periodicals may have formed the opinion that the subject of applied mathematics has been fully covered. As far as the industrial and military fields go, the editors believe that the text has provided excellent coverage, when the limitations of space are taken into consideration, but the business world provides a new and rich ground which is equally important. In the fields of insurance, accounting, finance, and statistics, mathematics plays its rôle in varied forms from simple arithmetic to the calculus.

This is particularly important during a period of transition from war to peace-time industrial reorganization. Despite the most careful planning, the transition from war to a peace-time basis promises to be an uncomfortable one for many. The man who equips himself to enter or re-enter the world of business will escape with a minimum of hardship. It is very possible that he will be the first to find new employment. Business offices will of necessity begin operations before the wheels in the reorganized plants begin to turn. Whereas the war worker who has been trained only in manufacture may be caught in the doldrums of industrial readjustment, the man who has something to offer to the business world may find himself in immediate demand.

The business mathematician should be one of these. To acquire the qualifications requires training as specialized as that which is needed for the industrial fields. While it is true that simple arithmetic takes care of many of the burdens imposed in business mathematics, in other situations the use of higher branches of mathematics is not only necessary but also expeditious. They provide short cuts to business problems, just as they effect time economies in the industrial fields.

There is no prescribed break-down in the field of business to which the pedants rigidly cling. Generally speaking, however, the four branches of insurance, accounting, finance, and statistics are accepted as convenient and logical divisions upon which to predicate any approach. When we enter the insurance field, we enter the realm of actuarial mathematics. While the study of actuarial mathematics is not so popular as other mathematical subjects, a number of important universities in the country have seen fit to make it an important if not a greatly publicized part of their curriculum. The subject of actuarial mathematics—as one may gather—provides the insurance man with the tools with which he calculates risks and arrives at premium values. In the rapidly-growing insurance business, there are boundless opportunities for the man who is expert in this field.

Accounting is entirely a matter of mathematics. It has many sub-divisions such as budgeting, business records, trade discounts, commissions, depreciation, leaseholds, etc. While arithmetic is able to answer some of the demands of the accountant, there is still need for recourse to other mathematics. Naturally, logarithms are extensively used.

In the business-statistics division, averages, medians, modes, trend graphs, frequency of occurrence, index numbers, and charts are studied. The statistician really has need for the more advanced branches of mathematics, and the calculus is frequently resorted to in his compilations. In the field of finance, we find a subject that has as much popular as specific interest. Its study is of interest not only for the man preparing for a financial career, but also to the countless millions who daily follow the newspaper records of the financial world. Most everybody has a personal interest in some stock or

bond, the value fluctuations of which are found on the financial pages. To many, these figures are just so much black ink on white paper. A knowledge of the fundamentals of finance helps in understanding the significance of the daily changes.

These facts are given to stress the importance of business-mathematics training, and to convince our readers that there are other fields to be conquered besides the "industrial and military applications".

R. S. K.

ABOUT OUR AUTHORS

WILLIAM W. MICHAEL has been with the California Institute of Technology since 1918 as associate professor of civil engineering. Prior to his appointment he was identified with a great number of private and public engineering projects which provided him with the practical background that has enriched his teaching. Even today he is engaged in many activities outside the academic world. Notable among these are his service as consulting engineer for the Mt. Palomar telescope project and his work as a supervisor of the Engineering, Science and Management War Training Program.

Professor Michael was born at Palatine Bridge, N. Y., in 1888. He attended Tufts College from which he obtained a bachelor of science degree in civil engineering in 1909. From 1909 to 1911, he worked for the City of New York on topographic surveys. In 1912 and 1913, he was a construction engineer for a nationally known construction company. The year 1914 saw him at Michigan Agricultural College where he instructed in the department of drawing and design. Following a period devoted to a private engineering practice, 1916–1918, he was appointed to the staff of the California Institute of Technology.

AS THE author of the article on radio, we offer George F. Maedel, the Chief Instructor of the New York School of the R.C.A. Institutes, an organization which specializes in the training of technicians for the radio industry.

Mr. Maedel first joined the staff of the R.C.A. Institutes in 1933 as the original instructor of a newly-formed mathematics department. Three years later, he was transferred to the radio frequency department, and in 1940 was appointed to the post of Chief Instructor.

Born in Brooklyn, New York, in 1903, Mr. Maedel studied at Columbia University. He earned the Bachelor of Arts degree in 1924 and his Electrical Engineer rating in 1926. While still at the university, he gained practical experience with the New York Edison Company, and after graduation joined the engineering staff of that utility. From 1927 to 1931, he was in the traffic-engineering department of the New York Telephone Company. Then followed a two-year period during which he formed and operated the Audio Products Engineering Company, which specialized in the installation of public-address systems. During his business career, Mr. Maedel found time for teaching. He was laboratory instructor in electrical engineering at Columbia University in 1927, and mathematics instructor in the evening classes at Pratt Institute from 1929 to 1933. He has written two textbooks on mathematics applied to radio which are now in use at the R.C.A. Institute.

· THE MATHEMATICS OF NAVIGATION ·

By William W. Michael, B.S. in C.E.

NAVIGATION is the science of determining position on the earth's surface. More commonly, the term, *navigation*, refers to the methods of determining the position of a ship or airplane or other *moving* object with respect to the earth's surface. The determination of position of *fixed* points is a more refined process and lies in the realm of surveying.

There are four methods which are commonly used in marine and aerial navigation. These are: *piloting*, the determination of position by reference to fixed landmarks or by soundings; *dead reckoning*, the determination of position by reference to the last known position and distances and directions traveled from that position; *radio navigation*, the determination of position by radio bearings and beams; *celestial navigation*, the determination of position by observations on the sun, moon, planets, and stars.

MEASURING POSITION AND DISTANCE

In order to have an absolute system of reference which will enable us to determine position anywhere on the earth's surface, we must introduce a system of coördinates. The system which has been universally adopted is that employing *longitude* and *latitude*. To define these terms, let us refer to Fig. 1.

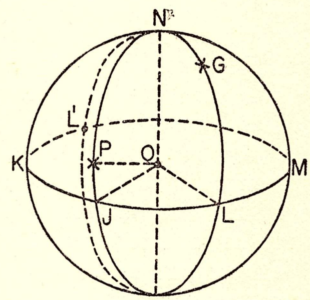


Fig. 1

Let O be the center of the earth, which we shall assume to be a sphere*. The line, NOS, represents the earth's axis, N being the North Pole and S the South Pole. The plane through O perpendicular to the axis and intersecting the sphere in the great circle, $KJLM$, is called the *equatorial plane* and the great circle, $KJLM$, is called the *equator*. Great circles passing through S and N are called *meridians*. Small circles which are formed by the intersection of planes parallel to the equatorial plane and the earth's surface are called *parallels of latitude*. The particular meridian which passes through Greenwich, England, is called

* Throughout this article, we shall assume the earth to be an exact sphere. Actually, the earth is an oblate spheroid whose polar diameter is about 27 miles shorter than its equatorial diameter. For our purposes, we can neglect the effect of the oblateness.

the *prime meridian*. If G represents Greenwich, then $NGLS$ is the prime meridian. We denote its intersection with the equator by L.

Now let us consider any fixed point, P, on the earth's surface. Let $NPJS$ be the meridian passing through P, J being the point where this meridian intersects the equator. We define the *longitude* of P to be the angle, LOJ, or the length of the equatorial arc, LJ, measured in degrees, minutes, and seconds. Longitude is measured from the prime meridian, which is taken as 0°, both to the east and to the west up to 180°. Thus, if L' is the point on the equator diametrically opposite to L, all points on the half great circle, $NL'S$, will have a longitude of 180°. All points lying west of the prime meridian and east of the 180° meridian will have *west longitude*; all points lying east of the prime meridian and west of the 180° meridian will have *east longitude*.

The *latitude* of a point, P, is defined to be the angle, JOP, for the arc, JP, measured in degrees, minutes, and seconds along the meridian. Latitude is said to be north if P lies north of the equator and south if P lies south of the equator. Thus, the equator is at latitude 0°, the North Pole at $+90°$ and the South Pole at $-90°$.

The difference of longitude between two points is the length in angular measure of the smaller equatorial arc intercepted by the meridians which pass through the two points. The difference in latitude of two points is the length in angular measure of an arc along a meridian intercepted by the parallels of latitude which pass through the two points.

Distance, direction, and departure

The unit of distance commonly used in navigation is the *nautical mile*. It is approximately equal to one minute (1′) of arc on the earth's equator. Its length is 6,080.27 feet, or about $\frac{8}{7}$ of the statute mile of 5,280 feet. A speed of one nautical mile per hour is called a *knot*. Thus, the distance a ship traverses in nautical miles is obtained by multiplying the mean speed in knots by the time traveled in hours.

The direction, or *bearing*, of a point with respect to an observer is the angle between the observer's meridian and the great circle passing through the point and the observer. This angle is always measured from the north, taken as 000°, clockwise through 360°, and is called the *azimuth*. Thus, a point whose bearing was east would have an azimuth of 090°; southwest, an azimuth of 225°, etc.

The *course* of a ship is defined as the direction of its line of travel. It is measured in degrees clockwise from the north point in the same manner as azimuth.

A line which makes equal angles with all of the meridians is called a *rhumb line*. In general, a rhumb line is a curve which spirals toward the poles. The exceptions to this are the parallels of latitude, which make an angle of 90° with each meridian, and the meridians, which make an angle of 0° with themselves. Thus, the parallels, equator,

and meridian are seen to be special cases of rhumb lines. Except in the case of the equator and the meridians, the rhumb line joining two points will not coincide with the great circle through the two points. Hence, in the case of great-circle sailing, it becomes necessary continually to change the course.

Above, we saw that one minute of longitude on the equator is equal to one nautical mile. It is apparent that, in higher latitudes, due to the convergence of the meridians toward the poles, one degree of longitude is less than one nautical mile. The resulting problem is, for a given latitude, to determine the length in nautical miles of one degree in longitude. To determine this, let us consider Fig. 2.

The difference in longitude between the two meridians, NQ and NP, is measured by the angle, ϕ. Since a dihedral angle is constant, ϕ is also equal to ϕ'. We are given the latitude, θ, of the parallel, ST, and we wish to know the length of ST in nautical miles. Now the length of PO is $\phi = \phi'$ nautical miles where ϕ and ϕ' are measured in minutes of arc.

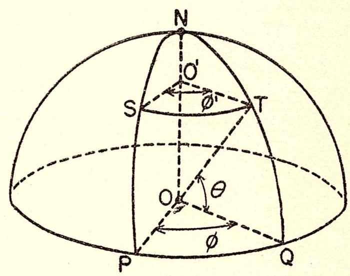


Fig. 2

If we let $r = OQ = OT$ be the radius of the earth, then

$$PQ = r\phi.$$

If we let $r' = O'T$ be the radius of the arc, ST, of the parallel of latitude, then

$$ST = r'\phi' = r'\phi.$$

But $\qquad r' = r \cos \theta;$

$\therefore \qquad ST = r\phi \cos \theta,$

or $\qquad ST = PQ \cos \theta.$

That is, the length of ST in nautical miles is equal to the length, PQ, in nautical miles times the cosine of the latitude.

This is the same as saying: the length, ST, in nautical miles equals the length, PQ, in minutes of longitude times the cosine of the latitude.

The length in nautical miles of an arc on a parallel is called the *departure*. It is clear that the angular measure is the same for all parallels, whereas the departure depends on the cosine of the latitude. This law can be restated as follows:

$$\text{departure} = (\text{difference in longitude}) \times (\text{cos latitude}).$$

The sailings

In setting the course for sailing, we may take our choice of several methods of calculating, depending upon the degree of accuracy which is necessary under the conditions imposed.

PLANE SAILING

In considering small portions of the earth's surface, we may assume without introducing appreciable error that the portion is a plane.

We must now find the relations between course, distance, difference of longitude and latitude, and departure under the above assumption.

Assume that a ship initially at *A* (Fig. 3) sets sail on a course, *C*, at a rate of *s* knots. *AN* represents the meridian through *A*. Our problem is to find the difference of latitude and the departure after a time of *T* hours. At the end of *T* hours, the ship will have traveled *sT* nautical miles (*AE* in Fig. 3) to the position, *E*. Now, if we assume this portion of the earth's surface to be a plane, we have the difference in latitude as *AD*, measured in nautical miles or minutes of arc. The departure is *DE*, measured in nautical miles. In the right triangle, *AED*, we have the following relations:

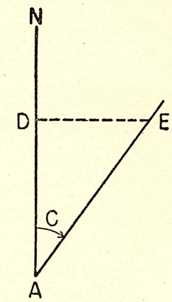


Fig. 3

$AD = AE \cos C$; *i.e.*, the difference in latitude equals the distance traveled times the cosine of the course.

$DE = AE \sin C$; *i.e.*, the departure equals the distance traveled times the sine of the course.

The two problems occurring most frequently in plane sailing are:

a Given the course and distance, find the difference of latitude and the departure.

b Given the difference of latitude and departure of the destination, find the course and distance.

Illustrative Problem A

A ship sails from a position of Lat. 20°00′00″ N, Long. 150°00′00″ W on a course of 120° at 10 knots. Find the difference of latitude and departure after 4 hours.

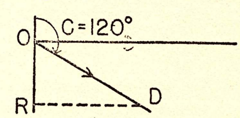


Fig. 4

The distance sailed, *OD* (Fig. 4), equals 10 knots×4 hours = 40 nautical miles.

The difference of latitude, $OR = OD \cos C = 40 \cos 120° = 40 \sin 30° = 40 \times 0.50000 = 20$ nautical miles.

The departure $= OD \cos 30° = 40 \times 0.86603 = 34.64$ miles*.

Illustrative Problem B

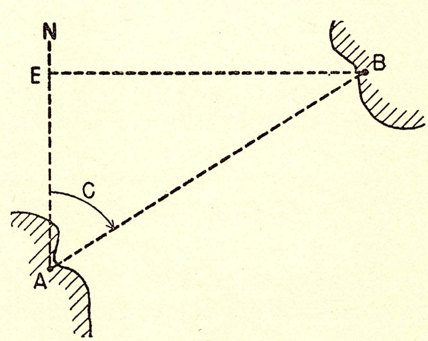


Fig. 5

The captain of a ship wishes to sail from port *A* (Lat. 37°40′00″ N, Long. 135°00′00″ W) to port *B* (Lat. 38°00′00″ N). The departure is known to be 30 miles. Treating the earth's surface as a plane, determine what course should be taken, and the distance from *A* to *B*.

The difference of latitude is 20′, or 20 nautical miles.

* Throughout this section, problems will be solved with the aid of tables of logarithms and trigonometrical functions.

In Fig. 5, $\tan C = \dfrac{EB}{AE} = \dfrac{\text{departure}}{\text{diff. in lat.}} = \dfrac{30}{20} = 1.500.$

Hence, C is 56°18′.

$$\text{Distance } AB = \frac{EB}{\sin C} = \frac{\text{departure}}{\sin C} = \frac{30}{0.83206} = 36.06 \text{ miles.}$$

TRAVERSE SAILING

Traverse sailing is composed of several rhumb-line tracks like those discussed under plane sailing. To find the final difference of latitude and departure after several courses have been traversed, we must take the algebraic sums of the separate courses and departures.

Illustrative Problem

A patrol ship determines its position to be Lat. 52°00′00″ N, Long. 140°00′00″ W at 12:00 NOON. If the following courses and rates are used, find the departure and difference of latitude at 10:00 P.M.

COURSE	RATE (*knots*)	TIME (*hours*)	COURSE	RATE (*knots*)	TIME (*hours*)
072°	10	3	180°	15	1
072°	12	1	270°	15	2
145°	12	2	270°	18	1

A tabular form of solution is expedient.

	COURSE	RATE (*knots*)	TIME (*hr.*)	DISTANCE (*naut. mi.*)	MILES MADE GOOD East	South	West	North
a	072°	10	3	42	39.95	0	0	12.98
	072°	12	1					
b	145°	12	2	24	13.77	19.66	0	0
c	180°	15	1	15	0	15	0	0
d	270°	15	2	48	0	0	48	0
	270°	18	1					
				TOTALS	53.72	34.66	48	12.98
					48.00	12.98		
				Net E.	5.72	Net S. 21.68		

Miles made East in $a = 42 \sin 72° = 39.95$.
Miles made East in $b = 24 \cos 55° = 13.77$.
Miles made North in $a = 42 \cos 72° = 12.98$.
Miles made South in $b = 24 \sin 55° = 19.66$.

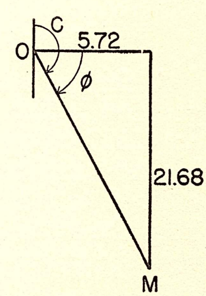


Fig. 6

Next, determine the course and distance made good:

In Fig. 6, $\tan \phi = \dfrac{21.68}{5.72} = 3.79$

$$\therefore \quad \phi = 75.2°$$

But $C = 90° + \phi = 165.2°$

$$\text{Distance} = OM = \frac{21.68}{\sin \phi}$$

$$= \frac{21.68}{0.9668} = 22.45 \text{ miles.}$$

PARALLEL SAILING

If the course is 090° or 270°, then the ship remains at a constant latitude. Thus, the distance made good will be all departure. Our problem is to find the difference of longitude in this case.

Illustrative Problem

A ship's initial position is Lat. 40°00′00″ N, Long. 130°00′00″ W. If the ship sets a course of 090° and its rate is 12 knots, what is its position after 5 hours?

$$\text{distance} = 12 \times 5 = 60 \text{ nautical miles.}$$

Since the course is due east, the distance is all departure.

By the formula,

$$\begin{array}{c}\text{departure in}\\ \text{nautical miles}\end{array} = \begin{array}{c}\text{difference of longi-}\\ \text{tude in minutes}\end{array} \times \cos \text{latitude.}$$

Hence, $$\text{difference of longitude} = \frac{60}{\cos 40^\circ} = 60 \times 1.30541$$
$$= 78.32', \text{ or } 1^\circ 18' 19''.$$
$$130^\circ 00' 00'' - 1^\circ 18' 19'' = 128^\circ 41' 41''$$

∴ The final position is: Lat. 40°00′00″ N, Long. 128°41′41″ W.

MIDDLE LATITUDE SAILING

The departure between two places of different latitude is not the same as the departure measured on the parallel of either place. It is found, however, that the true departure is approximately equal to the departure on a parallel half-way between the parallels passing through the two locations. In the case that the two positions under consideration are on opposite sides of the equator, it is then necessary to consider two distances and take the half-latitudes of each. The above assumption is fairly accurate provided that distances are below 250 miles and latitudes are less than 50°. There are two problems which commonly arise here:

a Given the longitude and latitude of the point of departure and the course and distance, find the point of arrival.

b Given the longitude and latitude of two points, find the course and distance between them.

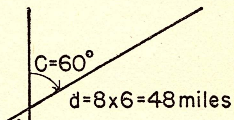


Fig. 7

Illustrative Problem A

A ship sets sail from a harbor (Fig. 7) whose position is Lat. 40°00′00″ N, Long. 70°00′00″ W, at 12:00 NOON and takes a course of 60°. It is steaming at 8 knots. This information is radioed to a submarine commander. He assumes that the ship will keep the same course and speed. What position will he take to intercept the steamer at 6:00 P.M.?

If l = difference in latitude

$$l = 48 \cos 60^\circ = 24'$$

∴ $L_2 = 40^\circ 24' \text{N} = \text{latitude at 6:00 P.M.}$

$$\text{Mean latitude} = \frac{L_1 + L_2}{2} = 40°12' \text{ N.}$$

$$\text{Departure} = 48 \sin 60° = 41.6 \text{ miles.}$$

$$\text{Difference in longitude} = \frac{\text{departure}}{\cos\text{ (mean lat.)}} = \frac{41.6}{\cos 40°12'} = \frac{41.6}{0.7638°} = 54.5'$$

$$\therefore \quad \lambda_2 = 69°5'30'' = \text{longitude at 6:00 P.M.}$$

$\therefore$ The commander will take his submarine to Lat. 40°24′ N, Long. 69°5′30″ W.

Illustrative Problem B

A ship sends an SOS giving her position as Lat. 40°24′ N, Long. 69°5′30″ W. What course should a coast-guard cutter whose position is Lat. 41°00′ N, Long. 69°29′.3 W take in order to reach the distressed ship? If the cutter's top speed is 16 knots, how long will it take to reach the ship? (See Fig. 8.)

Difference in latitude = 36′ or 36 miles

$$\text{Mean latitude} = 40°42' \text{ N} = \frac{L_1 + L_2}{2}.$$

$$\begin{aligned}\text{Departure} &= \text{difference in long.} \times \cos \text{mean lat.}\\ &= 23'.8 \times \cos 40°42'\\ &= 23'.8 \times 0.75813 = 18 \text{ miles.}\end{aligned}$$

$$\begin{aligned}\text{Course} = C &= 180° - \tan^{-1}(0.5)\\ &= 180° - 26.6° = 153.4°\end{aligned}$$

$$\text{Distance} = \frac{18}{\sin 26.6} = 40.3 \text{ miles.}$$

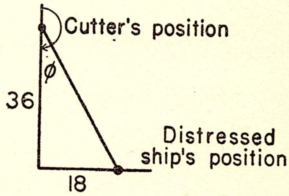


Fig. 8

At 16 knots, it will take the cutter $\frac{40.3}{16}$ or 2.52 hours to reach the ship.

DEAD RECKONING

Dead reckoning is the process by which a ship's position is brought up to date from the last known *position* (which was determined either by astronomical sights or shore sights). This process uses the course and speed of the ship together with any available information about currents and wind. Graphical methods and the method of middle-latitude sailing are most commonly employed, but the method of Mercator sailing, which will be explained later, is also used.

Illustrative Problem A

The following table gives the courses and mean speeds taken by a vessel whose position is Lat. 52°00′00″ N, Long. 140°00′00″ W:

Course	Speed (*knots*)	Hours	Course	Speed (*knots*)	Hours
072°	10	3	180°	15	1
072°	12	1	270°	15	2
145°	12	2	270°	18	1

(a) Find the position of the ship after 10 hours by the method of middle latitude sailing. (b) If there is a known current of set 060° (*set* is the direction of flow) and drift of 3 knots (*drift* is the speed of the current) in the entire localitv, what will be the true position of the ship?

This is the same as the example on page 773, from which we find:

5.72 miles are made East
and 21.68 miles are made South.

Hence, difference in latitude $=21'41''$.

$\therefore$ The new latitude $=(52°00'00'')-(21'41'')=51°38'19''$ N.

$$\text{The mean latitude} = \frac{(51°38'19'')+(52°00'00'')}{2} = 51°49'09''$$

$$\text{Hence, difference in longitude} = \frac{\text{dep.}}{\cos \text{lat.}} = \frac{5.72}{\cos (51°49'09'')} = 9'15''.$$

$\therefore$ The new longitude $=(140°00'00'')-9'15''=139°50'45''$ W.

Taking into account the current:

In the 10 hours, the ship would have drifted 30 miles toward 060°; *i.e.*, the effect of the current is to displace the ship

$30 \cos 60° = 15$ miles North
and $30 \sin 60° = 25.98$ miles East

These must be added to the net miles made good. We find that the distance made good is

$5.72+25.98=31.70$ miles East
and $21.68-15=\ 6.68$ miles South.

Using these new values, and proceeding in the same manner as before, we find that the true latitude is

$$52°00'00''-6'41''=51°53'19''.$$

$$\text{The mean latitude} = 51°56'40''.$$

$$\text{The difference in longitude} = \frac{31.70}{\cos (51°56'40'')} = \frac{31.70}{0.61635} = 51'24''.$$

$$\text{The true longitude} = 140°00'00''-51'24''=139°08'36'' \text{ W}.$$

Illustrative Problem B

A ship is 50 miles due south of her destination. If she is to arrive at her destination in 4 hours, what course and speed should she take if she must steam in a current of set 150°, drift 4 knots?

A graphical solution is recommended. (See Fig. 9.)

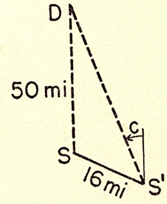


Fig. 9

In 4 hours, if the ship were stationary, it would have drifted to S'.

$\therefore$ The ship must sail the distance, $S'D$, and take the course, C.

TEST YOUR KNOWLEDGE OF SAILINGS WITH THESE EXERCISES

1 How much time elapses between 0440 and 1610 on the same day?*
2 A ship is sailing a course with a bearing of 194°. What angle does the ship's course make with South?
3 A ship travels from 28°30′ N, 19°10′ W to 30°08′ N, 20°12′ W. In nautical miles, what is (a) its departure; (b) its change in latitude; (c) the distance it has traveled?
4 A ship's position at 0830 is Lat. 48°15′, Long. 71°15′ W. She makes a course bearing 74° at a speed of 15 knots. Find her position at 1400.
5 Find the bearing of the ship's course in Fig. 9 (page 776), solving by plane trigonometry.
6 A ship sails a course bearing 85° at 12 knots for 5 hours. Then she alters her course, bearing 214° at 10 knots for 6 hours. What course should she then set to return to the starting point?

PILOTING AND RADIO DIRECTION FINDING

Piloting is the process of navigating a vessel which is in sight of land or other fixed points by taking the bearings and distance of these points. The easiest way to make the necessary computations is graphically.

For piloting in American coastal waters, the navigator has at his disposal extraordinarily detailed charts issued by the United States Government, showing lighthouses, beacons, buoys, church steeples, and other objects of reference. Coastal waters in other parts of the world have also been charted in varying degrees of detail.

Illustrative Examples

Assume that the chart of Fig. 10 represents that portion of the coastline along which we are sailing.

At 1400*, we sight the lighthouse, A, bearing at 050°. This means that at 1400 we are located somewhere along the line passing through A and bearing $180°+50°=230°$. This line is drawn on our chart and labeled with the time above the line and the bearing below.

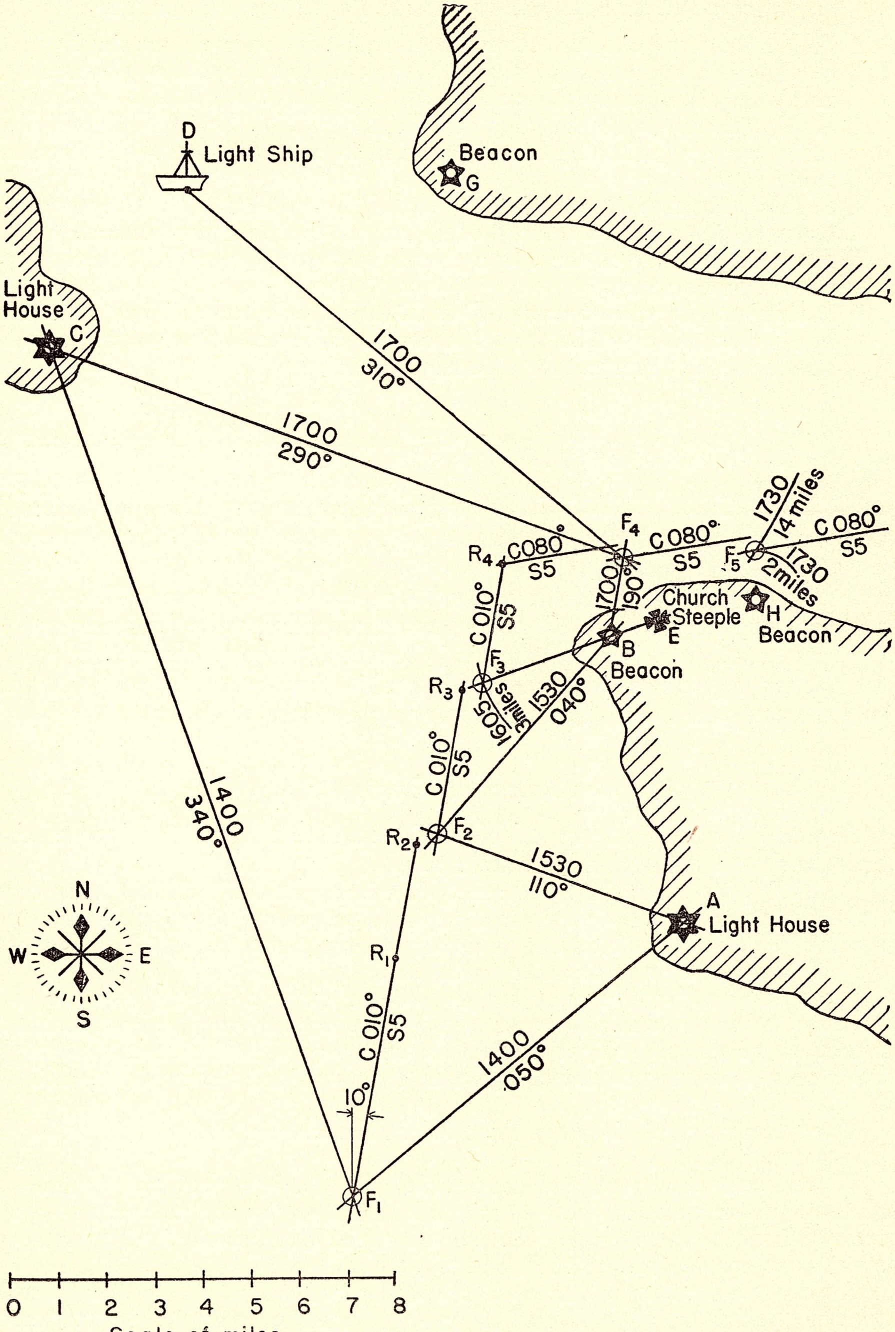
Since one bearing does not *fix* our position, it is necessary to have two bearings, a bearing and a distance, or two distances in order to fix a position. Hence, if we also sight light C at the same time and find that its bearing is 340°, then we are somewhere along a line through C and bearing $340°-180°=160°$.

The intersection of these two lines definitely locates the ship at F_1, at 1400. The point, F_1, is called a *fix*.

If we know our speed and course, we can plot it on the chart, and we shall know our position relative to the last fix, F_1, by the methods of dead reckoning explained on page 775. For example, if our course is 010° and our speed 5 knots, we shall be at R_1 at 1500.

At 1530, according to the dead reckoning, we are at R_2; we take simultaneous sights on lights A and B and find their bearings to be 110° and

* Ship's time is reckoned from 0000 midnight to 2400 midnight. Thus, 7:00 A.M. =0700, 10:00 A.M. = 1000, 1:30 P.M. =1330, 2:00 P.M. =1400, 5:05 P.M. =1705, 8:00 P.M. =2000, etc.



D
Light Ship
Beacon
G
Light House
C
1700
310°
1700
290°
1400
340°
F4
C 080°
S5
R4
C080°
S5
1730
14 miles
C 080°
S5
F5
1730
2 miles
1700
190°
Church Steeple
E
H
Beacon
B
Beacon
C 010°
S5
F3
R3
1605
3 miles
1530
040°
C 010°
S5
R2
F2
1530
110°
A
Light House
N
W
E
S
R1
C 010°
S5
1400
050°
10°
F1
0 1 2 3 4 5 6 7 8
Scale of miles

Fig. 10

40° respectively. These lines are drawn on the chart and labeled. We find that at 1530 we are actually at F_2 instead of at R_2. The discrepancy may be due to unknown currents or to error in the mean speed. We continue sailing on our course, but we now mark the track through F_2, abandoning the erroneous point, R_2.

At 1605, we notice that the beacon, B, and the church steeple, E, are in line. This places our position somewhere along a line drawn through B and E. Our position is presumably at R_3 but, measuring the distance to B by the range finder, or some other device, we find that the distance is 3 miles. A circle with center at B and a scale radius of 3 miles is drawn. The intersection of this circle with EB extended gives F_3, our position at 1605.

We continue on our course until 1635 and change course to 080°. We do this on the chart by extending the 010° line from F_3, the last fix, for a distance of $2\frac{1}{2}$ miles $\left(5 \text{ knots} \times 30 \text{ minutes} = 2\frac{1}{2} \text{ miles}\right)$. This brings us to R_4. Then, through R_4 we lay off the new course, labeling it $\frac{C\,080°}{S\,5}$; *i.e.*: course 080°, speed 5 knots. At 1700, a fix is obtained by our taking simultaneous bearings on the light ship, D, the beacon, B, and the lighthouse, C. This gives a good check of position. The three lines intersecting at the point, F_4, give a new fix. The course line is drawn through F_4 and the positions reckoned from the last fix, F_4, until a new fix is taken. Assume that at 1730 we check our distance from H and find it to be 2 miles. A simultaneous sight on the light ship, D, gives 14 miles. We can obtain a fix by plotting a circle with center at H and a radius of 2 miles, and a circle with a center at D and a radius of 14 miles. These circles are labeled $\frac{\text{time}}{\text{distance}}$ and their intersection gives the new fix, F_5.

This example shows most of the methods of locating a ship's position by sights on known objects. Two or more simultaneous bearings (*e.g.*, F_1, F_2, F_4) or a distance and direction fixed by alignment (*e.g.*, F_3), two alignments or two distances (*e.g.*, F_5) will, in most cases, fix the ship's position.

Sometimes it is not possible to take two simultaneous sights and time elapses between readings. In this case, what is called a *running fix* can be obtained.

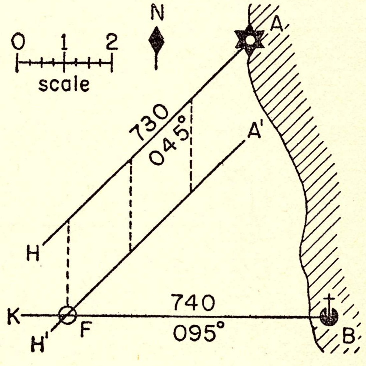


Fig. 11

Illustrative Example

Our vessel is taking a course of 180° and making 12 knots. At 730 we take a sight on light A (Fig. 11) and find it bears 045°. This places our position somewhere along the line, AH. At 740, a sight on the church, B, gives a bearing of 095°: *i.e.*, we are somewhere along the line, KB. Now, we assume that we know our course 180° and speed 12 knots with fair accuracy for the 10-minute period from 730 to 740. In this period, any point on AH will have shifted

a distance $\frac{1}{6}\times 12=2$ miles due south. Thus, if the ship were at some point on AH at 730, then it must be at some point on $A'H'$ at 740. $A'H'$ is AH translated 2 miles south. Therefore, at 740 we are on both $H'A'$ and KB, that is, at F. The point, F, is called a running fix.

Radio bearings

There are two methods used in determining position by radio. In one method, a radio compass station on land listens to the signal of a ship and takes its bearing. The station then reports this bearing to the ship. In the second method, the ship uses a direction finder to take the bearing of a signal broadcast by some land compass station. These land stations are listed in the same manner as beacon lights and lighthouses. They are recognized by distinctive signals which are listed in the *light lists* published by the government.

Not only direction, but also distance can be obtained from radio compass stations. This is done by sending out either an air or a submarine sound signal simultaneously with the radio signal. The ship notes the difference of time between the reception of the radio signal and the reception of the sound or submarine signal; knowing the latter's velocity, one can compute the distance. These methods are not very accurate because of variation in the speed of sound.

Illustrative Example

A vessel on a 95° course is in a heavy fog off shore and wishes a radio fix. The direction finder picks up a signal bearing 30° from ship's head. This signal is being sent out simultaneously with a submarine sonic signal. It takes 5 seconds for the submarine signal to arrive after reception of the radio signal. If the speed of the submarine signal is known to be 4400 ft. per sec., what is the bearing of the radio station and its distance?

The station will bear $95°+30°$, or 125°. In 5 seconds, the signal will have traveled 22000 ft. or $\frac{22000}{6080}=3.62$ nautical miles; *i.e.*, the distance from the radio compass station is 3.62 miles.

TEST YOUR KNOWLEDGE OF PILOTING WITH THESE EXERCISES

7 A pilot is following a course bearing 345° at 10 knots. A lighthouse bears 270° at a distance 12 miles from the ship. After $1\frac{1}{2}$ hours, how far is the ship from the lighthouse?

8 A pilot is navigating a channel with a lighthouse on one side and a beacon on the other, 1 mile due west of the lighthouse. The bearing of the beacon is 14° and of the lighthouse 54°. What course should the ship steer to pass midway between the two objects? Solve graphically to the nearest degree.

SOME FORMULAS OF SPHERICAL TRIGONOMETRY

Spherical triangles are formed by the arcs of great circles on the same sphere. Both sides and angles are measured in degrees, minutes, and seconds. An angle is measured by the plane angle between lines tangent to the great circles at their intersection. A side is measured by the angle it subtends at the center of the sphere.

Right spherical triangle

In Fig. 12, we have a spherical triangle, ABC, with sides a, b, c. Let O-ABC be a spherical pyramid of the sphere with center at O and with radius r. It is clear from the fact that all radii of a sphere are equal that

$$OA = OB = OC = r.$$

By convention, the right angle of the spherical triangle will be taken at C. Thus, $C = 90°$. By definition, if the tangents, k and l, to angle C are drawn, the angle between them will be 90°. In other words, angle $LCK = 90°$. We wish now to show that right angle LCK is a plane angle of the dihedral angle, OC. This will be shown in detail.

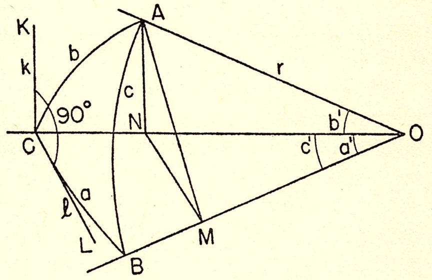


Fig. 12

The tangents to a and b at their point of intersection, C, are each perpendicular to line OC because (by plane geometry in separate planes OAC and OBC) a line in a plane tangent to a circle is perpendicular to the radius, OC, at the point of contact.

Consequently, as we see easily, the angle, LCK, is a plane angle (in fact, a right plane angle) of the dihedral angle, A-OC-B, with the faces, AOC and BOC. By solid geometry, the dihedral angle is measured by its plane angle. Since this plane angle is a right angle, the faces of the dihedral angles—or planes AOC and BOC—are perpendicular to each other.

From the point, A, a plane, MNA, is constructed perpendicular to OB, cutting OB in point M and OC in point N. Now, OM is perpendicular to MA and MN, since (by solid geometry) a line perpendicular to a plane is perpendicular to every line in the plane passing through its foot. Thus, angles OMA and OMN (as well as AMB and NMB) are right angles. Consequently, angle AMN is a plane angle of the dihedral angle, OB.

Since a spherical angle (B) is measured by the dihedral angle formed by the intersecting planes of the two great circles (AB and BC), angle B then equals plane angle AMN of dihedral angle OB.

Furthermore, the plane, AMN, is perpendicular to plane BOC, since a plane perpendicular to a line lying in a second plane is perpendicular to this second plane.

It is also known that, in addition to the fact that plane AMN is perpendicular to OCB, plane OAN is perpendicular to plane OBC, since these two planes form a right dihedral angle for right spherical angle C.

Since plane AMN is perpendicular to plane OCB and plane OAN is perpendicular to plane OCB, AN is perpendicular to the plane, OBC, because (by

solid geometry) if a plane (OCB) is perpendicular to each of two intersecting planes (OAN and ANM) it is perpendicular to their line of intersection (AN).

Since AN is perpendicular to plane OBC, it is perpendicular to all lines (ON, MN, NC) in the plane passing through its foot.

In other words, angles ANO and ANM are right angles.

Additionally, it has been shown that angles OMN and OMA are right angles.

Therefore, we have the right triangles, AOM, NOM, NOA, besides MNA.

Angles NOM, NOA, and AOM—*i.e.*, a', b', c' of these right triangles—are central angles of the arcs, a, b, c; or, in other words, the face angles of the trihedral angle, O-ABC, (or the central angles) are equivalent to the arcs, a, b, c, of the great circles.

Moreover, each of these face angles now can be expressed as a trigonometric function of, or a trigonometric relationship among, the corresponding parts of its own right triangle.

If we now take Fig. 12, we can express the sides of the right triangle as trigonometric functions of the radius of the sphere (r) and the face angles, a, b, c.

Therefore (by trigonometry of the right triangle), the following expressions result:

$$\frac{NA}{r}=\sin b \qquad \frac{MA}{r}=\sin c \qquad \frac{OM}{r}=\cos c \qquad \frac{ON}{r}=\cos b \qquad \textbf{Ia}$$

$$NA=r\sin b \qquad MA=r\sin c \qquad OM=r\cos c \qquad ON=r\cos b \qquad \textbf{Ib}$$

$$\frac{MN}{OM}=\tan a=\frac{MN}{r\cos c} \qquad \frac{MN}{ON}=\sin a=\frac{MN}{r\cos b} \qquad \textbf{IIa}$$

or

$$MN=r\cos c\tan a=r\cos b\sin a \qquad \textbf{IIb}$$

Fig. 13 gives the labels of the sides of the right triangles of the spherical pyramid. These relationships of Ib and IIb will be utilized in the development of formulas for the solution of right spherical triangles. The derivation proceeds as follows:

From the triangle AMN, where the plane angle, AMN, of the dihedral angle, OB, is equal to the spherical angle, B (previously shown), these relationships result:

$$\sin B=\frac{r\sin b}{r\sin c}=\frac{\sin b}{\sin c}. \qquad \textbf{IIIa}$$

$$\cos B=\frac{r\tan a\cos c}{r\sin c}=\frac{\tan a\cos c}{\sin c}. \qquad \textbf{IVa}$$

$$\tan B=\frac{r\sin b}{r\sin a\cos b}=\frac{\sin b}{\sin a\cos b}. \qquad \textbf{Va}$$

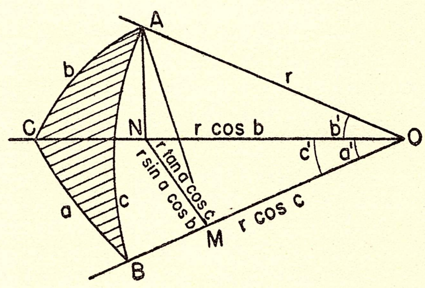


Fig. 13

Since the r's cancel each other, these expressions may be written as:

$$\sin B\sin c=\sin b. \qquad \textbf{IIIb}$$

$$\frac{\cos c}{\sin c}=\cot c.$$

$$\cos B = \tan a \cot c. \qquad \text{IVb}$$

$$\tan B \sin a \cos b = \sin b.$$

$$\sin a = \frac{\sin b}{\cos b} \, \frac{1}{\tan B}.$$

$$\sin a = \tan b \cos B. \qquad \text{Vb}$$

$$\tan a \cos c = \sin a \cos b.$$

$$\cos c = \sin a \cos b \, \frac{1}{\tan a}.$$

$$= \sin a \cos b \, \frac{\cos a}{\sin a}.$$

$$\cos c = \cos b \cos a. \qquad \text{VI}$$

Had a plane been passed through B perpendicular to the edge, OA, these formulas similarly would result:

$$\sin a = \sin c \sin A. \qquad \text{VII}$$

$$\cos A = \tan b \cot c. \qquad \text{VIII}$$

$$\sin b = \tan a \cot A. \qquad \text{IXa}$$

The reader will observe that formulas VII, VIII, and IXa are obtained by interchanging a and b in formulas IIIb, IVb, and Vb.

There are three other useful formulas which we shall derive from these.

$$\cot A = \frac{\sin b}{\tan a}. \qquad \text{IXb}$$

$$\cot B = \frac{\sin a}{\tan b}. \qquad \text{IXc}$$

Multiplying IXb by IXc, we have

$$\cot A \cot B = \frac{\sin b \sin a}{\tan a \tan b} = \cos a \cos b.$$

Substituting for the right-hand member its value from VI,

$$\cot A \cot B = \cos c. \qquad \text{X}$$

From IVb, $\cos B = \tan a \cot c = \dfrac{\sin a}{\cos a} \cot c$.

From VI and VII $= \dfrac{\sin c \sin A}{\dfrac{\cos c}{\cos b}} \cot c$

$$= \tan c \cos b \cot c \sin A$$

or $\cos B = \cos b \sin A$. XI

From VIII, $\cos A = \tan b \cot c$

$$= \frac{\sin b}{\cos b} \, \frac{\cos c}{\sin c}$$

From IIIb and VI, $= \dfrac{\sin c \sin B}{\cos b} \, \dfrac{\cos a \cos b}{\sin c}$

or $\cos A = \sin B \cos a$. XII

If, in a right spherical triangle, there are two or three right angles, these formulas are not necessary. If the reader will draw

such figures, he will see that they can be solved by inspection. A right triangle with one side equal to 90° is a *quadrantal spherical triangle.* If the right triangle contains two right angles, the sides opposite these angles are each equal to 90°—one quadrant—and therefore equal to each other. Such a triangle is an *isosceles triangle* as well as a quadrantal triangle.

In plane trigonometry, if the terminal side falls in the first, second, third, or fourth quadrant, the angle is of that quadrant. Such is the case in spherical trigonometry.

Here are two helpful rules concerning right triangles. These are useful in that they serve as a check upon one's work.

Rule I—The oblique angle and its opposite side are of the same quadrant.

Rule II—(a) If the hypotenuse of a right spherical triangle is less than 90°, the two legs are of the same quadrant. (b) If the hypotenuse is greater than 90°, one leg is of the second quadrant.

The reader will benefit if he will sketch several right spherical triangles of different sizes illustrating these rules.

Napier's rules

The formulas derived above need not be remembered. Though it is possible for one to pick the formulas one needs depending upon the information given and to letter one's figure to fit the formulas, a far simpler method was devised by the mathematician, Napier.

Given the right spherical triangle, ABC, with sides a, b, and c, the parts of the right spherical triangle (excluding the right angle, C) arranged in order, we may place them around a circle as in Fig. 14. These parts are known as *circular parts.* The three parts forming the hypotenuse of the right triangle (A, c, B) are always written as their complements ($90°-A$, $90°-c$, $90°-B$) to make the rules work. Instead of writing out these complements, we may

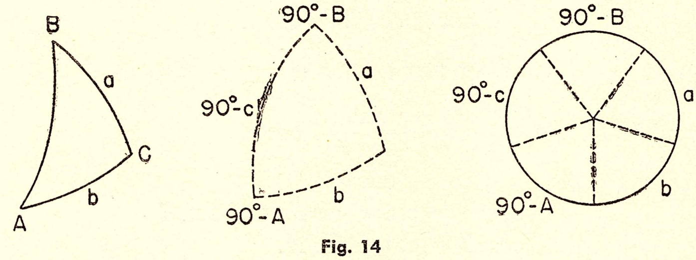


Fig. 14

invent our own notation for these three parts, such as boxing them, underlining them, barring them, etc. In this text, two bars will be used; thus:

$$\overline{\overline{A}}=90°-A,\ \overline{\overline{B}}=90°-B,\ \overline{\overline{c}}=90°-c.$$

For any one of these circular parts (one of which we shall now say is fixed), the two parts lying next to it, one on each side, are called *adjacent parts*; any two other parts are known as *opposite parts* for the fixed circular part.

Thus, if b is fixed, the adjacent parts are a and $\overline{\overline{A}}$; the opposite parts are $\overline{\overline{c}}$ and $\overline{\overline{B}}$.

Rule I—The sine of any circular part is equal to the product of the tangents of the two adjacent parts.

Rule II—The sine of any circular part is equal to the product of the cosines of the two opposite parts.

From Napier's rules, the formulas for the right spherical triangle can be derived. (The rules are a compendium, not a proof, of the formulas.) They enable us to determine all the parts of a right spherical triangle, if we are given any two of them.

Illustrative Example

Given the spherical triangle with known parts b and A, find the formulas for the other parts.

$$\textbf{a} \quad \sin \overline{\overline{B}} = \cos b \cos \overline{\overline{A}} \text{ (By Rule I)}$$
$$\sin (90°-B) = \cos b \cos (90°-A)$$
$$\textbf{b} \quad \cos B = \cos b \sin A$$

Now that B is known, we can say by Rule II

$$\sin \overline{\overline{c}} = \tan \overline{\overline{A}} \tan \overline{\overline{B}}$$
$$\sin (90°-c) = \tan (90°-A) \tan (90°-B)$$
$$\textbf{c} \quad \cos c = \cot A \cot B$$

The last unknown part, a, may be found by either rule.

By Rule I

$$\sin a = \tan b \cot \overline{\overline{B}}$$
$$\textbf{d} \quad \sin a = \tan b \cot B$$

By Rule II

$$\sin a = \cos \overline{\overline{A}} \cos \overline{\overline{c}}$$
$$\sin a = \sin A \sin c$$

The general spherical triangle

To find the relation between the sides and the angles of a general spherical triangle, consider Fig. 15.

Take the radius, OC = unity; thus:

$$x = \cos a$$
$$y = \sin a \cos B$$
$$z = \sin a \sin B$$

Also:

$$x' = \cos b$$
$$y' = \sin b \cos A$$
$$z' = \sin b \sin A.$$

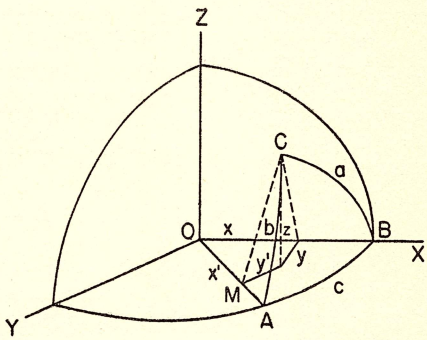


Fig. 15

Looking at Fig. 16 showing the XY-plane, we have the following relations:

$$x = x' \cos c + y' \sin c$$
$$y = x' \sin c - y' \cos c$$
$$z = z.$$

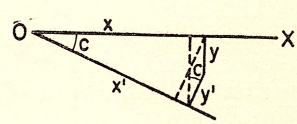


Fig. 16

Equating, we have:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \qquad \text{XIII}$$
$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A \qquad \text{XIV}$$
$$\sin a \sin B = \sin b \sin A. \qquad \text{XV}$$

TEST YOUR KNOWLEDGE OF SPHERICAL TRIGONOMETRY

9 Write a formula giving the cosine of the hypotenuse (c) of a right spherical triangle in terms of the trigonometric functions of the adjacent angles (B and C), using Napier's rules.

10 What is the air distance between El Paso, Texas (Lat. 31°45′ N, Long. 106°30′ W) and Quito, Ecuador, on the equator in Long. 78°30′? Express the result both in nautical miles and in statute miles.

11 A plane stopped at Galapagos on the equator at Long. 92°. It set its course for Havana, Lat. 23°09′ N, Long. 82°21′ W by middle latitude sailing. How far was the bearing of the course in error?

12 In the oblique spherical triangle, ABC, the side, a, is 42°; b is 117°; and the angle, C, between them is 35°. Find the side, c. (Hint: Apply formula XIII, using c, a, b to designate the sides, a, b, c, of the formula.)

13 Land's End, England, has the position Lat. 50°04′ N, Long. 5°45′ W. Portsmouth, N. H., has the position Lat. 43°05′ N, Long. 70°44′ W. What is the distance along the great-circle course from Portsmouth to Land's End?

THE MERCATOR CHART AND GREAT-CIRCLE SAILING

From the examples on pages 771-776, it is seen that a chart on which a rhumb line is a straight line is highly desirable. Such a chart is called a *Mercator Chart* and is obtained as follows:

First, we wrap a cylinder about a sphere in such a manner that it is tangent to the sphere along the equator (Fig. 17). We then pass planes through the earth's axis, NOS. These planes will cut the sphere along meridians and the cylinder along straight lines perpendicular to the equator. Thus, the meridian, NKS, corresponds to the element, RKT, on the cylinder, and every meridian will correspond to an element of the cylinder. Next, consider the parallel, ULV, whose latitude is θ. This line will be "projected" onto the cylinder into a line, $U'\,L'\,V'$, which is parallel to the equator and a distance, y, above the equator, where

$$y=\log\left[\tan\left(45°+\frac{\theta}{2}\right)\right].$$

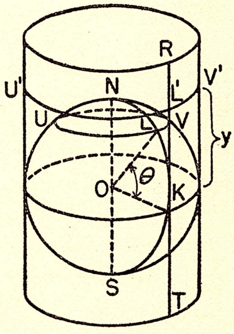


Fig. 17

If we cut the cylinder along an element and roll it out, we shall have a plane chart, such that any rhumb line on the earth appears as a straight line on the chart. When the earth's features are projected on such a chart, we obtain a Mercator Chart. The scale of longitude is the same throughout the chart, but the scale of latitude varies with the latitude. Tables have been made from which these scales can be obtained. If we desired to sail from point A to point B (Fig. 18), we could join A and B with a straight line, and this would be the rhumb line connecting A and B. The rhumb course between two points such as A and B can be taken directly off of the chart, here the angle, θ.

The shortest path joining two points on the earth's surface is the great-circle path. Except in the cases mentioned under distance (page 770), the great-circle path differs from the rhumb line. On

the Mercator Chart (Fig. 18), the great-circle path joining A and B is represented by a curved line. In the northern hemisphere, this falls to the north of the rhumb line; in the southern hemisphere, it falls to the south. If we are given the longitude and latitude of A and B and wish to sail a great-circle track between them, we must determine the longitude and latitude of several points on the path and plot them on the Mercator Chart. If we draw a smooth curve through these points, we can then find our courses from the chart.

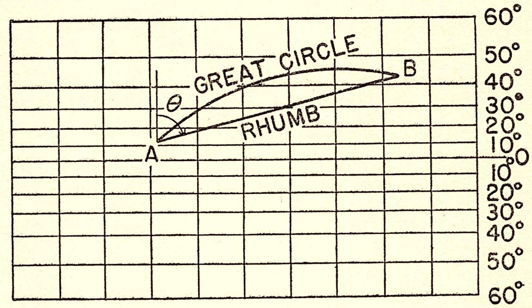


Fig. 18

Illustrative Problem

We wish to sail the great-circle route between New York (Lat. 40°40′ N, Long. 74° W) and the Straits of Gibraltar (Lat. 36° N, and Long. 5° 42′ W). Find the track and compute the courses and distance.

First, construct the spherical triangle, NAB (Fig. 19).

N is the North Pole
A is New York
B is Gibraltar

The side, AN, is equal to the co-latitude of $A = 90° - 40°40' = 49°20'$. NB = co-latitude of $B = 90° - 36° = 54°$. The angle, N, is equal to the difference in longitude:

$$= 74° - 5°42' = 68°18'.$$

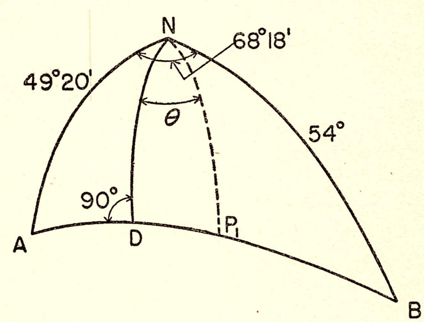


Fig. 19

To find the distance, AB, we employ formula XIII:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

We let $a = \widehat{AB}$; $b = \widehat{AN}$; $c = \widehat{BN}$; and $A = N$.

Hence: $\cos(\text{dist.}) = \cos 49°20' \cos 54° + \sin 49°20' \sin 54° \cos 68°18'$
$= (0.65166)(0.58779) + (0.75851)(0.80902)(0.36975)$
$= 0.38304 + 0.22689 = 0.60993$
$\therefore$ dist. $= 52°35'$.

To convert this to nautical miles, we need only multiply the 52° by 60 and add the 35′, since our path is a great circle. This gives the distance

$$\widehat{AB} = 3155 \text{ miles.}$$

Next, we can compute the initial course by using formula XV; *viz.*,

$$\sin A = \frac{\sin a}{\sin b} \sin B.$$

We substitute:

$$a = 54° = \overset{\frown}{NB}$$
$$b = 52°35' = \overset{\frown}{AB}$$
$$B = 68°18' = N.$$

$$\text{Hence, } \sin A = \frac{\sin 54°}{\sin 52°35'} \sin 68°18' = \frac{(0.80902)}{(0.79229)}(0.92913) = 0.94644$$

$$\therefore \quad A = 71°10'. \qquad \therefore \quad \text{Initial Course} = 71°10'.$$

The final course, B, can be found in a similar manner. We now have the initial course and the distance, but we must compute the longitude and the latitude of several points on the great-circle path in order to determine intermediate courses. To do this, let us drop a perpendicular from N to AB. Call D its intersection with AB*. Consider the right triangle, AND. We know angle A and side AN: hence, by Napier's rules, we can find angle AND and side ND.

$$\begin{aligned} \sin \overset{\frown}{ND} &= \cos (90° - 49°20') \cos (90° - 71°10') \\ &= \cos 40°40' \cos 18°50' \\ &= (0.75851)\ (0.94646) = 0.71790 \end{aligned}$$

$$\therefore \quad \overset{\frown}{ND} = 45°53' \text{—}i.e., \text{ the latitude of } D \text{ is } 90° - (45°53'), \text{ or } 44°07'.$$

Also, by Napier's rules:

$$\begin{aligned} \sin (90° - \angle AND) &= \tan (90° - \overset{\frown}{AN}) \tan \overset{\frown}{ND} \\ &= \tan 40°40' \tan 45°53' \\ &= (0.85912)\ (1.03126) = 0.88598 \end{aligned}$$

$$\therefore \quad 90° - \angle AND = 62°22' \qquad \therefore \quad \angle AND = 27°38\ .$$

Therefore, the longitude of D is $74° - 27°38' = 46°22'$.

We can now proceed to determine as many points as needed on the great circle. Lay off an arbitrary angle, θ, of (say) 10°. Draw NP_1 and solve triangle NDP_1 for NP_1.

By Napier's rules:

$$\begin{aligned} \sin (90° - \theta) &= \tan \overset{\frown}{ND} \tan (90° - \overset{\frown}{NP_1}) \\ \therefore \quad \tan (90° - \overset{\frown}{NP_1}) &= \cot ND \sin (90° - \theta) \\ &= \cot 45°53' \sin 80° \\ \tan (\text{Lat. } P_1) &= (0.96969)\ (0.98481) = 0.95496 \\ \therefore \quad \text{Lat. } P_1 &= 43°41'. \end{aligned}$$

Thus, the longitude and the latitude of P_1 are known:

$$\text{Long. } P_1 = \text{Long. } D - 10° = 36°20' \text{ W}$$
$$\text{Lat. } P_1 = 43°41' \text{ N}$$

It will be noticed that we have also the longitude and the latitude of a point, P_1', 10° west of D; *viz.*,

$$\text{Long. } P_1' = \text{Long. } D + 10° = 56°20' \text{ W}$$
$$\text{Lat. } P_1' = 43°41' \text{ N.}$$

In a similar manner, we may take $\theta = 20°$ and find points P_2 and P_2' on the great circle.

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* Point D is called the vertex of the great circle. It may be that D will not fall between A and B, but will have to be placed on the great-circle track extended. This would make no difference in the computations.

This process is repeated, using arbitrary increments in angle, until the longitude and the latitude of enough points on the track have been determined to allow its being plotted on the Mercator Chart. With the great-circle track plotted on the chart, we can join several points on it by rhumb lines. The ship does not actually sail the great circle, but sails a series of rhumb lines which approximate the great circle. These rhumb lines may be either chords of the great circle or tangents to the great circle, as shown in Fig. 20.

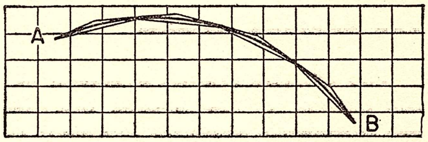


Fig. 20

COMPOSITE SAILING

Sometimes it is undesirable to sail a single great-circle track because of ice or other dangers to navigation. In this case, the track can be composed of more than one great circle together with several arbitrary rhumb lines. This type of sailing is called *composite sailing.*

TEST YOUR KNOWLEDGE OF GREAT-CIRCLE SAILING

14 On a certain Mercator-projection map of the world, the scale of miles to the inch is 500 at the equator. How many miles to the inch at Lat. 60°?

15 Point *A* has Lat. 40° N, Long. 80° W; point *B* has Lat. 42° N, Long. 75° W. What is the error computing the distance between *A* and *B* by middle latitude sailing?

16 Halifax, Nova Scotia, has the position, Lat. 44°40′ N, Long. 63°35′ W; Vladivostok, in Siberia, Lat. 43°11′ N, Long. 131°53′ E. A plane flies the great circle between them. How far from the pole, in degrees of latitude, does the course pass?

ASTRONOMICAL DEFINITIONS

When one, at night, looks into the sky and watches the stars rise, set, and move about, one can imagine them to be set upon the interior surface of a cosmic sphere which completely hems in the earth. The center of this *celestial sphere* is thought of as the place where the observer stands and watches the sphere revolve about him. The celestial sphere is thought of as indefinitely large, so that all terrestrial and even planetary motions shrink to the infinitesimal in comparison.

Defining a point

Just as in analytic geometry a point may be defined by two numbers (coördinates) which represent the distance of the point from the *X*- and *Y*-axes, a corresponding system has been devised by astronomers. The celestial sphere is thought of as covered with imaginary circles that intersect one another at right angles, the intersection defining a point on the sphere with reference to some fixed elements. These circles are similar to those of latitude and longitude lines on the earth. In this elementary work, two systems of coördinates will be employed: the *horizon system* and the *equinoctial system.* The

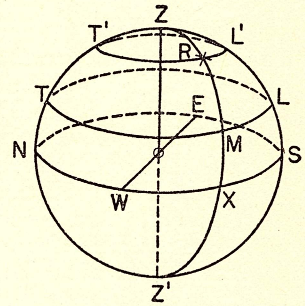
first of these is attached to the observer and the second to the celestial sphere. The two systems of coördinates are therefore in a state of diurnal rotation with respect to one another.

The horizon system

The horizon system is shown in Fig. 21. The earth is a small sphere at the center. A line joining the observer to the center of the earth defines the observer's vertical; extended upwards, this line cuts the ideal celestial sphere in the point directly overhead. This point is called the *zenith* (Z in Fig. 21); the (invisible) point directly opposite on the celestial sphere is the *nadir* (Z' in Fig. 21). The observer's vertical thus determines his zenith; it is the same line whose inclination to the plane of the equator determines the observer's terrestrial latitude.

The observer is standing on some particular terrestrial meridian, determining his longitude. The plane of this meridian (extended indefinitely) cuts the celestial sphere in a great circle called the observer's *meridian*.

The great circle on the celestial sphere at a quadrant's distance from the zenith (*i.e.*, the great circle lying in the plane perpendicular to the observer's vertical, ZZ') is called the *horizon* (NESW). This ideal line differs from the *visible horizon*, which is frequently jagged and irregular, with natural and man-made contours. The small circles parallel to the great circle of the horizon are called *almucantars* (as TML).



The Horizon System
Fig. 21

The great circles on the celestial sphere which pass through the zenith (and the nadir) are called *vertical circles*. Thus, ZXZ' in Fig. 21 is a vertical circle (only one-half of which is shown in Fig. 21). The particular vertical circle which passes through the north and the south points of the horizon ($ZNZ'S$) is the meridian, since it lies in the plane of the observer's terrestial meridian. The vertical circle at right angles to the plane of the meridian (through the points, Z, E, Z', and W, in Fig. 21) is called the *prime vertical.*

A point in the heavens, or a celestial object, is located with reference to the horizon system by means of the two coördinates, *altitude* and *azimuth.* The altitude is the angle of elevation, as viewed by the observer, above the horizon. The azimuth is the angle between the vertical circle on which the point or object lies and the meridian.

In Fig. 21, R is a point on the celestial sphere whose altitude is measured by the arc, RX, along the vertical circle, ZRX, and whose azimuth is measured by the arc, $NESX$, along the horizon, $NESW$. The object, R, might be estimated to have an altitude of 60° and the azimuth of 230° (which may be written as S50°W).

The equinoctial system

If the earth were stationary as well as flat, the horizon system might be ample. However, the earth rotates on its axis, carrying with it the horizon system, and complicates for the observer the motion of the heavenly bodies.

Depending upon one's position on the earth, some stars rise and set; others never rise and never set.

The north star, or pole star, is one of the very few stars which always appear in practically the same place in the heavens for all times of the day, for all seasons of the year. This is because Polaris happens to lie close to the axis of rotation (that is, the extension of the line joining the North Pole to the South Pole) of the earth, and of the apparent rotation of the celestial sphere.

The system of celestial coördinates which is fixed among the fixed stars (that is, with respect to which the horizon system rotates) is the *equinoctial* system of coördinates, so called after its equator, the *equinoctial line.*

The celestial sphere

The celestial sphere, with the equinoctial as well as the horizon system of coördinates, is shown in Fig. 22. The celestial North Pole is at *C.N.P.*, the celestial South Pole at *C.S.P.* The equinoctial line, generally called in astronomy the *celestial equator*, is *RWR'E.* The celestial equator is at a quadrant's distance from either pole. The meridians of this system are called *hour circles*; they are great circles passing through the celestial poles. Thus, the hour circle of the star, *ST.*, in Fig. 22 is the great circle, *C.N.P.-ST.-C.S.P.*

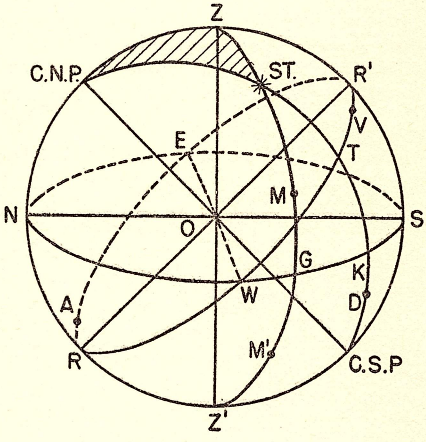


Fig. 22

The angle between the hour circle of a star and a certain circle used as a zero hour circle is called (not the celestial longitude, but) the *right ascension* of the star. The right ascension is equal to the dihedral angle between the hour circle and the zero circle, or to the arc along the equator between them. Thus, the right ascension of the star, *ST.*, in Fig. 22 is the arc *VT.* (The hour circle through the zero point, *V*, is not shown in Fig. 22.)

The angular elevation of a star (viewed from the center of the celestial sphere) above the plane of the celestial equator is called (not the celestial latitude, but) the *declination* of the star. It is measured by the arc along the hour circle between star and celestial equator. Thus the declination of *ST.* in Fig. 22 is *ST.-T.*

Right ascension

Right ascension is measured in hours, minutes, and seconds from $0^h0^m0^s$ at the zero point clear around the celestial equator, reaching 24^h at the zero point again. One hour of right ascension thus equals a dihedral angle of 15°. Declination is reckoned positive and negative from the equator, ranging from +90° at the celestial North Pole to 0° at the equator and −90° at the celestial South Pole.

A celestial object which has its right ascension and declination given is thereby definitely located with reference to the system of equinoctial coordinates (and therefore with reference to the system of fixed stars).

We saw previously that an object which has its altitude and azimuth

given is definitely located with respect to the horizon system of coördinates, that is, with reference to the observer's horizon and meridian (*NESW* and *NZSZ'* in Fig. 22).

DIURNAL MOTION

Obviously, a point which remains fixed with respect to one system will be in a state of rotation with respect to the other system. Thus, the star Arcturus, has the (virtually) constant "celestial position", *R.A.* (right ascension), 14^h13^m, δ (declination) 19°30′. It therefore revolves, rising and setting with respect to the horizon. The zenith, on the other hand, remains fixed in the horizon system; it therefore lies now on one, now on another, hour circle of the equinoctial system. The moon is an example of an object which remains fixed in neither system. About once a month, it completes a revolution in right ascension; it shares also the (apparent) rotation of the celestial sphere. Its motion with respect to the equinoctial system of coördinates (that is, its motion in right ascension and declination) is its true orbit motion; its apparent motion with respect to the horizon system of coördinates is a composite of its motion with respect to the equinoctial system and the relative motion of the two coördinate systems.

SOLAR MOTION

The motion of the sun with respect to the horizon system is composed of the relative motion of the sun and the earth plus the relative rotation of the coördinate systems. The relative motion of the sun and the earth lies in a plane (called the *ecliptic plane*) which defines a great circle on the celestial sphere, the *ecliptic circle*. The ecliptic circle intersects the celestial equator at two points, defining the *vernal equinox* and the *autumnal equinox*. The two planes are inclined 23°27′. The sun appears to revolve once around the ecliptic circle in a tropical year of 365.2422 days. At the moment when the sun is in both planes, it lies on both the ecliptic circle and the celestial equator. At that time, day and night are of equal length; hence the name, equinoctial circle, for the celestial equator. The vernal equinox is chosen as the zero point in reckoning right ascension.

ELEVATION OF POLE

Notice that the declination of the observer's zenith is equal to his latitude. This is because the observer's vertical line, connecting him with the center of the earth, passes through his zenith, also, by its inclination to the plane of the equator, fixes his latitude. From Fig. 22, we see that the altitude of the pole, the angle *C.N.P.-O-N*, is equal to the angle, *ZOR'*, and hence also equals the latitude. Thus, the latitude of any point on the earth is the altitude above the horizon of the celestial pole, as viewed from that point. This explains the classical use of the pole star (which lies within 2° of the celestial North Pole) as the mariner's guide to his latitude.

It is clear that Fig. 22 shows the relationships of the horizon coördinate system with the equinoctial coördinate system for an observer in a north latitude of somewhere around 45°. For an observer in any other latitude, the inclination of the zenith above the plane of the equator will be different, and a diagram of the configuration will look substantially different from Fig. 22. For an observer at the terrestrial North Pole, the celestial North Pole lies at the zenith. (For this exceptional case obviously the two systems of coördinates

coincide. The whole celestial sphere revolves around the zenith and any given fixed star revolves along an almucantar, never rising and never setting.) For an observer at the equator, the zenith lies on the celestial equator, and the celestial poles lie on the north and south horizon points.

Solar time and sidereal time

We have seen that the zenith, the point on the celestial sphere directly above the observer, is a point with continually changing hour angle. Its declination is constantly the same for a stationary observer, being equal to the observer's latitude; but its right ascension varies as the observer's meridian sweeps across the celestial sphere.

The observer's meridian passes through the zenith. It is a great circle through the poles. The hour circles are also great circles through the poles. The meridian is a circle of the horizon system; the hour circles are circles of the equinoctial system. The former is attached to the observer's position on the earth; the latter to the system of the fixed stars. About a line joining the poles as axis, the observer's meridian is constantly rotating with respect to the hour circles, coinciding now with one, now with another, of them. The rate of rotation is exactly constant.

The number of solar days in a year is $365\frac{1}{4}$. This is the number of times the sun rises. Since the earth revolves once about the sun during this period, the earth in a year actually rotates once more about its axis than the number of times the sun rises. The number of *sidereal days* in a year is, therefore, $366\frac{1}{4}$. The sidereal day is defined as the period of rotation of the earth with respect to the fixed stars (Latin *sideris*, of the star). It is $\frac{365.24}{366.24}=0.9972$ mean solar days, or 23 hours 56 minutes, in length. The plane of the meridian rotates through 24^h of right ascension exactly once each sidereal day.

THE HOUR ANGLE

The angle between the plane of the observer's meridian and the plane of the hour circle through any celestial object is the object's *hour angle*. The hour angle is also the arc (measured in hours) along the celestial equator between the observer's meridian and the hour circle.

For a fixed observer on the earth, then, the hour angle of a given star increases at the rate of 360° every 23 hours 56 minutes of mean solar time. At the same moment of time, the hour angle of a given star is different for observers at different longitudes (*i.e.*, on different meridians).

At any given point on the earth, the same star rises at the same sidereal time each night. It therefore rises 4 minutes earlier each night by mean solar time. A star's rising continually gains 4 minutes a day, so that, at the end of a year, it has gained one complete revolution over the sun.

Thus, the change of the hour angle of a star measures the change of sidereal time at a given place. When the hour circle in question is the zero hour circle, the hour angle itself is defined as the sidereal time at the given place on the earth. Thus, local sidereal time at any place is the hour angle of the vernal equinox at that place equal to the hour angle of any star with right ascension of 0^h.

LOCAL SOLAR TIME

The earth's orbit about the sun is slightly elliptical and the sun does not traverse with perfect uniformity its apparent angular motion along the ecliptic, nor in right ascension. We imagine an ideal sun whose angular motion in right ascension is exactly uniform, and which has the same period of revolution as the real sun. We call this ideal object the *mean sun*. *Mean solar time* in any place is the hour angle of the mean sun at that place. Since local sidereal time is the hour angle of the vernal equinox, it is clear that local sidereal time is equal to local mean solar time at the moment when the mean sun is at the vernal equinox. When the mean sun is at the autumnal equinox, the mean solar time differs 12^h from sidereal time. The mean sun loses about 4^m (actually $3^m56.6^s$) of right ascension daily to make up its complete revolution of 24^h ($=1440^m$) of right ascension in the year of about 365 days.

Mean solar time as defined has its zero at noon; we add 12 hours to mean solar time to obtain *local civil time*, having its zero at midnight. Local civil time varies with longitude; for convenience, we have divided the earth into zones about 15° wide in longitude, and adopted an average time throughout each zone called *standard time*. This is the time we read on clocks. Obviously, if a clock is carried from one time zone into another, it will read one or more hours different from the standard time in the new zone. A clock which agrees with the standard time at Greenwich, England (Greenwich Civil Time) will differ from standard time at any other zone a number of hours equal to the number of zones removed from Greenwich. A clock carrying Greenwich Civil Time will differ from local civil time (mean solar time plus 12^h) by an amount exactly equal to the longitude of the place east or west of Greenwich (reckoning the longitude in hours, one hour to each 15° of longitude). This fact provides one way by which the navigator can determine his longitude. Time of sunrise, or time when sun crosses the meridian, can be found from the almanac in local civil time; the ship's chronometer gives Greenwich Civil Time at the moment the occurrence is observed. The difference is the ship's longitude.

In astronomy and navigation, the astronomical triangle is of fundamental importance. In Fig. 22, it is given by *C.N.P.-Z-ST*. If three parts of the triangle be given, the other three parts may be determined. Applications of this triangle will follow in the next section.

TEST YOUR KNOWLEDGE OF CELESTIAL COÖRDINATES

17 A ship is in longitude 136°45′ East of Greenwich. At Greenwich midnight ($0^h0^m0^s$ Greenwich Civil Time), what is the local civil time at the ship?

18 A mariner finds from the *Nautical Almanac* that on August 28, 1943, Polaris has its upper culmination (upper crossing of the meridian) at 3^h24^m past midnight. He observes the altitude of Polaris at that time to be 47°18′. If the declination of Polaris is 89°00′, what is his latitude?

19 The right ascension of the star, Pollux, in angular measure is 115°27′30″. Express this in hours, minutes, and seconds of right ascension.

20 At Greenwich Civil Noon (12^h0^m) on September 4, 1943, the hour angle of the vernal equinox at the meridian of Greenwich is 162°44′. What is the sidereal time in New York (Long. 73°57′ West of Greenwich) at the same moment?

21 On September 1, 1943, at $2^h34^m02^s$ G. C. T. as given by the ship's chronometer, the ship is in longitude 46°18′ West of Greenwich. From the *Nautical Almanac*, it is found that the Greenwich hour angle of the star Vega (its hour angle at the meridian of Greenwich) is 99°15′. What is the hour angle of Vega at the ship?

22 On December 25, 1943, the star Pollux will cross the meridian of Greenwich at $1^h31^m16^s$ after midnight. At what time will it cross the meridian of New York (Long. 74°00′ W)? Express the result in Standard Time for New York.

23 If the mean sun crosses the vernal equinox at 0^h0^m Greenwich Sidereal Time on March 23, what is the right ascension of the mean sun at Greenwich Civil Noon on April 23?

POSITION BY CELESTIAL OBSERVATIONS

The methods of astronomical navigation are similar to the methods of piloting in that a line of position is obtained by taking a sight on a known object. In the case of piloting, the line of position is obtained by taking a sight on a known landmark, whereas, in celestial navigation, the line of position is obtained indirectly by taking a sight on the sun, moon, or stars. As in piloting, it takes two astronomical lines of position to determine a fix. If the two lines of position cannot be obtained simultaneously, it is then necessary to take a running fix.

Let us consider the case where two lines of position can be obtained simultaneously. In order to obtain a fix, we shall assume that we are in possession of the following information:

a Our approximate longitude and latitude, as obtained by dead reckoning from the last fix.

b The Greenwich Civil Time, as given by the ship's chronometer.

c The observed altitudes of two stars*.
(Assume that these altitudes were taken simultaneously.)

d The declinations of the stars as obtained from the *Nautical Almanac*.

We shall proceed to compute our exact position as follows:

In Fig. 23, let *NSZ* be the astronomical triangle. *N* is the elevated pole, *Z* the zenith, and *S* the observed star. Let *t* be the angle between the hour circle of the star, *SN*, and the meridian of the observer, *NZ*. We can now supply the following information: we take our assumed latitude obtained from dead reckoning and subtract it from 90°. This gives us the co-latitude, or side *NZ*. We take the star's declination from the *Nautical Almanac* and subtract it from 90°. This gives the polar distance, *SN*.

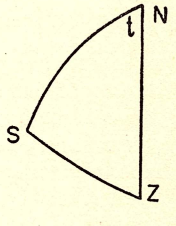


Fig. 23

Next, consider Fig. 24. This is a projection of the earth on the equator. *N* is the pole; *NG* is the meridian of Greenwich. *NO* is the meridian of the observer; *NS* is the projected hour angle of the star, *S*. From our dead reckoning position we know the approximate longitude, λ. Our chronometer gives the Greenwich Civil Time. We find the Greenwich hour angle, ϕ, of the star, *S*,

* The method of obtaining these altitudes is by sextant sights. A description of this instrument and its use is beyond the scope of this article. We suggest also that the reader consult other sources for discussion of the corrections to observed altitude and chronometer readings.

corresponding to the Greenwich Civil Time of the observation by using the *Nautical Almanac* or the *Air Almanac*. The local hour angle of the star is $t=\phi-\lambda$.

We now have two sides and the included angle in the triangle, NSZ. We wish to compute the zenith distance, SZ, of the star and the azimuth, Z.

This can be done by formula XIII:

$$\cos SZ = \cos NZ \cos NS + \sin NZ \sin NS \cos t.$$

Z can then be obtained from formula XV:

$$\sin Z = \frac{\sin t}{\sin SZ} \sin SN.$$

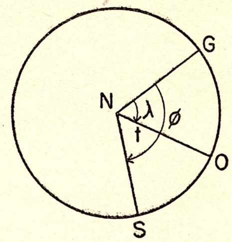


Fig. 24

The value of SZ obtained from formula XIII we shall subtract from 90° to give the *computed altitude* of S. Call this value H_c. We shall call the corrected observed altitude H_o. Now, if D is our position by dead reckoning (Fig. 25) at the time the altitude observations were made, we draw a line through D having the bearing, Z; *i.e.*, equal to the computed azimuth. The star, S, can have an altitude H_c only if the observer lies on the line, CC', which passes through D at right angles to the line of azimuth, SF (actually CC' is a short arc of the terrestrial circle through D about the point at whose zenith the star is located at the time of observation). Consequently, the star, S, can have the altitude H_o only if the observer is located on the line, KK', also perpendicular to SF, but displaced from CC' a number of nautical miles equal to the number of minutes of arc in H_c-H_o. Since H_o is the observed altitude, the ship lies somewhere on the line, KK'. If H_o is less than H_c, then the distance (H_c-H_o) is measured from D along SD *away from* S. If H_o is greater than H_c, then (H_c-H_o) is measured from D *toward* S. If the same process is now repeated with the second star, we shall get another line of position. The intersection of these two lines gives the position of the ship.

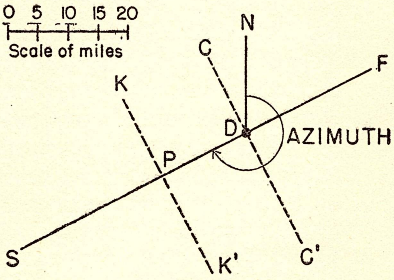


Fig. 25

This method of locating the ship's position, using a star's accurately observable altitude in connection with its computed azimuth (a slight error in which has little effect on the result) and altitude from the D.R. position to correct the latter, has been used for over a century in place of earlier more laborious methods. It was introduced by Sumner in 1837 and the lines of position derived by the method are called Sumner lines. This method is suitable for daytime observation, provided the sun and the moon are both visible. Right ascension and declination of these bodies for any time can be found in the *ephemeris**. In case the two observations are not quite simultaneous, the ship's position is located from the lines of position by means of a running fix (See page 779).

* An ephemeris is a table of celestial positions of a body for a series of advance dates. Ephemerides of the sun, moon, planets, and stars are included in the *Nautical Almanac*, annual publication of the United States Naval Observatory for mariners. Similar, but much condensed, data are given for the sun, moon, and planets in *American Air Almanac*.

Illustrative Example

At 2000 ship's time on January 1, 1943, the dead reckoning position was found to be Lat. 52°00′ N and Long. 45°00′ W. Simultaneous observations were taken on Aldebaran and Alpheratz. The corrected observed altitudes were found to be H_o at Aldebaran: 50°11′; and H_o at Alpheratz: 49°46′. The azimuth of Aldebaran was noted to be in the second quadrant East and that of Alpheratz in the second quadrant West. If the G.C.T. of the chronometer was 23^h22^m at the time of reading, what was the ship's position?

First, construct the astronomical triangle for Aldebaran (See Fig. 26).

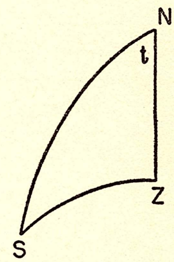


Fig. 26

$$\widehat{NZ} \text{ is } 90° - 52° = 38°.$$

From the *Nautical Almanac*, we find the declination of Aldebaran to be 16°24′ N and its G.H.A. at 23^h22^m to be 23°04′.

$$\therefore \quad \widehat{NS} = 90° - 16°24' = 73°36'.$$

The assumed longitude is 45° W.

$$\therefore \quad t = 45° - 23°04' = 21°56'.$$

Applying XIII,

$$\begin{aligned}\cos SZ &= \cos 38° \cos 73°36' + \sin 38° \sin 73°36' \cos 21°56' \\ &= (0.78801)\,(0.28226) + (0.61566)\,(0.95934)\,(0.92762) \\ &= 0.22242 + 0.54782 = 0.77025\end{aligned}$$

$$\therefore \quad SZ = 39°37'.$$

Now, applying XV, we have

$$\sin Z = \sin \widehat{NS}\,\frac{\sin t}{\sin SZ} = \frac{\sin 73°36' \sin 21°56'}{\sin 39°37'}$$

$$= \frac{(0.95934)\,(0.37353)}{(0.63774)} = 0.56187$$

Z is the angle of the second quadrant whose sine is 0.56187. Thus

$$Z = 145°49' \text{ E} = \text{Azimuth}.$$

Similarly, we compute SZ and Z for the Alpheratz. Again,

$$NZ \text{ is } 90° - 52° = 38°.$$

The declination of Alpheratz is 28°46′ N. Then

$$NS = 90° - 28°46' = 61°14'.$$

The G.H.A. of Alpheratz is 89°52′. The longitude is 45°.

$$\therefore \quad t = 44°52'.$$

By XIII,

$$\begin{aligned}\cos \widehat{SZ} &= \cos \widehat{NZ} \cos \widehat{SN} + \sin \widehat{NZ} \sin \widehat{SN} \cos t \\ &= \cos 38° \cos 61°14' + \sin 38° \sin 61°14' \cos 44°52' \\ &= (0.78801)\,(0.48137) + (0.61566)\,(0.87652)\,(0.70865) \\ &= 0.37932 + 0.38242 = 0.76174\end{aligned}$$

$$\therefore \quad SZ = 40°23'.$$

By XV,

$$\sin Z = \frac{\sin t}{\sin SZ} \sin SN = \frac{\sin 44°52' \sin 61°14'}{\sin 40°23'}$$

$$= \frac{(0.70556)\,(0.87652)}{(0.64788)} = 0.95454$$

Z is the angle of the second quadrant whose sine is 0.95454. Thus,

$$Z = 107°21' \text{ W}$$
$$\therefore \quad \text{Azimuth} = 360° - 107°21' = 252°39'.$$

(Here we take $360° - Z$ as the azimuth, because Z represents an angle measured *West.* The star had already crossed the meridian at the time of the observation, as signalled by the fact that its hour angle *exceeded* the longitude.)

Now, referring to Fig. 27, which represents the chart, O is the dead reckoning position at 2000. Through O, we lay off two lines, one having the computed bearing of Aldebaran at the time of the observation (*viz.*, 146°); the other having the azimuth of Alpheratz (*viz.*, 253°). Next, we take the difference, $H_o - H_c$, for Aldebaran:

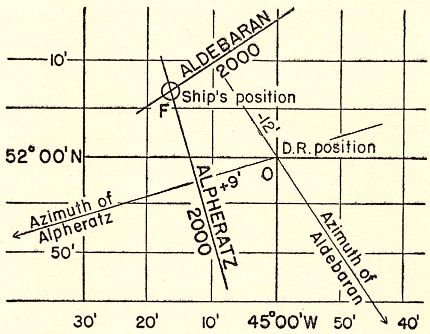


Fig. 27

$$H_o = 50°11'$$
$$H_c = 90° - 39°37' = 50°23'$$
$$H_o - H_c = -12'.$$

Since H_o is less than H_c, this distance will be *away from* the subastral point. We lay off 12 miles and get the Aldebaran line of position marked $\frac{\text{Aldebaran}}{2000}$. Similarly, we construct the line for Alpheratz:

$$H_o = 49°46'$$
$$H_c = 90° - 40°23' = 49°37'$$
$$H_o - H_c = +9'.$$

Since H_o is greater than H_c, the difference, 9′, is laid off *toward* the direction of Alpheratz. The Alpheratz position-line is plotted and the fix is at F, the intersection of the two lines.

Solution of the spherical triangle by the navigator has been rendered unnecessary by the publication a few years ago of the Navy document, H.O. 214, which gives in a series of volumes, solutions of the astronomical triangle for each degree of hour angle, each degree of latitude, each 30′ of declination. These immeasurably simplify the plotting of Sumner lines, for the choice of the ship's D. R. position as a starting point is purely conventional: we may as well choose any nearby position. By choosing as the starting position one on an even degree of latitude, and in such a longitude that the hour circle of the observed star comes out to an even degree, we can enter the tables with tabular values of the arguments, latitude, and hour angle. No interpolation is required except for the declination. The tabular functions are the computed altitude and azimuth desired.

Some old mariners, schooled in the plotting of Sumner lines, against the D.R. position, refuse to believe in the arbitrary initial position as a substitute. In using the H.O. 214 tables, they insist on

interpolating for all three arguments of the exact D.R. position. Of course, the computed altitude and azimuth they arrive at are very nearly as accurate as that obtained the easier way.

TEST YOUR KNOWLEDGE OF LINES OF POSITION

24 The star, Markab, will be at the zenith for an observer at Lat. 15° N, Long. 140° W at 0^h23^m local time. Find 4 points on the earth's surface at which the altitude of this star above the horizon will be 75° at that time (2 points on the 140th meridian, 2 points on the 15th parallel).

AERIAL NAVIGATION

The same methods used in marine navigation are also employed in aerial navigation—*viz.*, piloting, dead reckoning, celestial navigation, and radio. In the case of aircraft navigation, radio has taken a position of importance which greatly exceeds that of radio in marine navigation, but, in time of war, military necessity at times causes radio to be silenced and aircraft must fall back on the methods used in marine navigation. In most cases, the changes necessary for application of marine methods to aircraft are small. The methods and concepts of celestial navigation are the same in both cases and further discussion of astronomical methods is not necessary. The idea and practice of dead reckoning is the same, save for the effect of the wind. In marine navigation, we had to deal with currents and their effect on computed position. In the air, the wind plays the same rôle as do currents in the sea. The pilot must allow for the direction and the velocity of the wind in all of his dead-reckoning computations. The following will illustrate:

By *air-speed* is meant the speed of the airplane relative to the air. By *ground-speed* is meant the speed of the plane relative to the ground. It is apparent that ground velocity is the vector sum of air velocity and wind velocity.

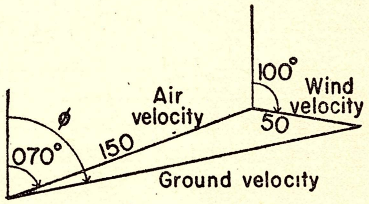


Fig. 28

Illustrative Problem A

An airplane on a course of 070° has an air speed of 150 m.p.h. If the wind is blowing 50 miles an hour from 280°, what course is the plane making relative to the ground?

The angle ø is the course made good and is measured as in Fig. 28.

Illustrative Problem B

A pilot wishes to make good a course of 120°. What course should he fly if his air speed is 100 m.p.h. and the wind is blowing 10 m.p.h. from 020°?

Fig. 29 is the aerial triangle for this problem. The course to be made good is first plotted as a line bearing 120°. The wind vector is next plotted as a line of length 10, bearing 200° (=180°+020°). From the end of the vector as center, we describe an arc with radius 100 intersecting the 120° vector in a point. Connecting the last two points gives us the course to be headed, bearing the angle designated as F in the figure.

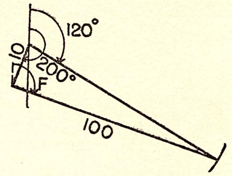


Fig. 29

The method of air piloting differs somewhat from marine piloting. The usual procedure is as follows: The pilot obtains a chart of the section between his base and the destination. He marks a straight line connecting these two points and then examines this path carefully to see whether there are any natural obstacles. For example, it would be unwise to fly over a lofty mountain range unless the plane had a high ceiling. It is also unwise for land planes to fly over extended stretches of water. The line of flight must be adjusted so that all such obstacles are avoided. Having fixed a suitable path, the pilot then studies the various landmarks along the route. He picks out such objects as water-towers, railroads, grain elevators, etc., which are distinct enough to identify from the air. Then he computes the approximate distances of these landmarks and, knowing his approximate speed, makes a table of the probable times at which he should sight them. The pilot should also study landmarks to both sides of his projected line of flight in order to locate himself should he be forced off his course. Charts used for air-navigation contain the location of beacons, direction lights, and emergency landing fields. The location of all these should be carefully noted.

This type of piloting can be done without any instruments whatsoever; for most small planes, a compass is available. This is a useful auxiliary for staying on the projected course. So far as is possible, it is wise to fly from landmark to landmark, always keeping a known point in view. The compass should not be relied on completely as the plane can easily get off course due to the wind. The compass alone would not record this fact.

TEST YOUR KNOWLEDGE OF THE WIND TRIANGLE

25 Calculate the bearing of the course made good by the plane in Fig. 28.

26 An air pilot heads his plane on a bearing of 90° at an air speed of 200 miles an hour. He finds from sighting back to an object over which he passed that he is making good a course of 97°. Assuming the wind is from the NW, determine the wind velocity.

27 A plane is flying in a steady wind. Heading a course of 270° at 225 miles per hour, the pilot finds the course made good to be 264°. The pilot now heads 180° and finds he makes good a course 186°. What is the wind velocity? (Solve graphically.)

THE APPLICATION OF MATHEMATICS TO RADIO

By George F. Maedel, A.B., E.E.

A PREVIOUS section was devoted to the applications of mathematics to electricity. This section is devoted to applications of mathematics to radio. Why should this distinction be observed? The flow of current, or electrons, and the mathematical explanations of this "fluid" flow had been associated with wired electrical *power* circuits for many years before wireless or radio circuits were known to the layman, or even to the power electrical engineers. Therefore, all mathematical explanations of electrical circuits were based on several assumptions; first, that the circuits were complete *wire* circuits; second, that the "fluid" flowed *from* the positive terminal of the generator, through the load circuit *to* the negative side of the generator; third, that Ohm's law and Kirchhoff's laws were always and continuously applicable.

The transformer is the major, if not the only, power-circuit element that transfers electrical energy from one circuit to another without a wire connection between the circuits. The radiation of energy through extended space, such as the phenomenon of transmission from an antenna, and the flow of electrons from the *filament* of a vacuum tube to the *plate* constitute rather violent digressions from power engineering concepts. Therefore, *power* engineers and *radio* engineers are rather distinct groups and while both groups use the same mathematics (all the mathematics they are able to assimilate), their literature and their technical meetings are usually separated from one another.

The radio technicians may be divided into two groups, those who operate or maintain radio equipment and those who design this equipment. The first group, which must know how radio circuits function, finds that a mathematical education through intermediate algebra and trigonometry is sufficient. The second group must have a thorough familiarity with the calculus and, if possible, with such graduate mathematics as hyperbolic functions, Bessel's functions, vector analysis, differential equations, and the theory of functions of real and complex variables.

ALGEBRA IN RADIO

Since resistors are used everywhere in radio circuits, the physical size is usually made as small as feasible to minimize the space required and the weight. However, these resistors must dissipate energy in the form of heat and are rated, therefore, according to their power-dissipating capabilities as well as according to their resistance.

Resistance

The value of resistance required and the current flowing are determined by the nature of the radio circuit. Therefore, the size in power-dissipating capabilities, of the resistor is calculated from the formula:

$$\text{power, } P = I^2R. \qquad \text{I}$$

Of course, algebra permits us to calculate the current flow allowable in a given resistor by the formula,

$$I = \sqrt{\frac{P}{R}}, \qquad \text{II}$$

and to calculate the resistance, if we are given the power dissipation and the current, by the formula,

$$R = \frac{P}{I^2}. \qquad \text{III}$$

By this time, the reader is well aware of the fact that resistors impede the flow of current in electrical circuits and that this impedance is designated by the symbol, R; is measured in *ohms*; and is numerically given by the equation,

$$R\text{(ohms)} = E \text{ (volts)} \div I \text{ (amperes)}. \qquad \text{IV}$$

(See pages 709 to 714.)

Capacitance

Condensers are equally ubiquitous in radio circuits and also impede the flow of current. This impedance, measured in *ohms*, is given by the formula,

$$X_C = \frac{1}{2\pi fC}. \qquad \text{V}$$

The impedance of a condenser is not so easily determined as that of a resistor, nor is its effect on the circuit so easily predicted. This statement may be clarified as soon as we review the inductance.

Inductance

Inductors (usually coils of wire) possess the property called *inductance* and impede the flow of current. This impedance, measured in *ohms*, is given by the formula,

$$X_L = 2\pi fL \qquad \text{VI}$$

To predict the electrical response of a given or proposed radio circuit, we should be able to set up mathematical equations so that mathematical operations may be performed. The radio technician must be thoroughly conversant with the right triangle and its attendant trigonometry and with the processes of *complex numbers*. The right triangle should be set on Cartesian coördinate axes as shown in Fig. 30c, where $X = X_L - X_C = 4.33 - 0.33$.

It is obvious (from Fig. 30a) that resistance, R, is measured horizontally (abscissa distances) and is measured to the right when R is positive and to the left when R is negative. Incidentally, the vacuum tube is the best-known means of obtaining *negative* resistance. This property of the vacuum tube

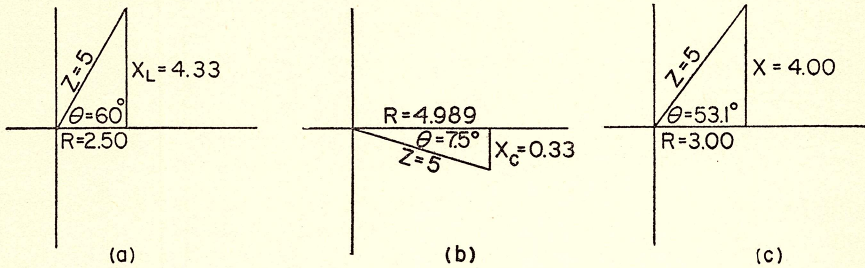


Fig. 30

in a properly designed circuit is all-important, for it is one of the major sources of sine waves at frequencies higher than are obtainable from rotating machinery. Fig. 30a also shows that reactance is measured vertically, and, since inductive reactance is indicated, it is measured upward. Fig. 30b shows that capacitive reactance is negative and is measured downward, while Fig. 30c shows that the *total* reactance is the *difference* of the inductive and the capacitive reactances. This difference is measured upward if the inductive reactance is larger, and downward if the capacitive reactance is larger.

All three figures indicate that the total impedance is represented by the hypotenuse of the triangle and the phase angle by the angle between the resistance and the total impedance. The algebraic or analytic representation of these graphs is to use complex numbers. As stated on page 365, mathematicians use i to represent the vertical side, whereas engineers use j. The analytic expressions to permit the use of algebra and trigonometry in conjunction with Fig. 30 are as follows:

$$\underset{\text{rectangular form}}{R+jX} = \underset{\text{trigonometric form}}{Z(\cos\theta+j\sin\theta)} = \underset{\text{exponential form}}{Z\varepsilon^{j\theta}} = \underset{\text{polar form}}{Z\angle\theta} \qquad \text{VII}$$

$$Z=\sqrt{R^2+X^2} \qquad \theta=\tan^{-1}\frac{X}{R}$$

The graphs of Fig. 30, and the complex-number equations associated with it, are used again and again in the analysis of series and parallel circuits of R, L, and C in both transmitters and receivers. Our interest in these circuits is usually confined to their properties at or near resonance.

Circuits

Circuits of R, L, and C in series are common in radio; the antenna circuit of a long-wave transmitter or receiver is an example. From Fig. 31, it is evident that the antenna circuit,

where R = wire plus radiation resistance,
C = antenna capacity, antenna wire to ground,
and L = circuit inductance (shown at A),

is a series circuit and that the current that flows is given by the formula:

$$I=\frac{E}{\sqrt{R^2+X^2}}=\frac{E}{\sqrt{R^2+(X_L-X_C)^2}}=\frac{E}{\sqrt{R^2+\left(2\pi fL-\frac{1}{2\pi fC}\right)^2}}. \qquad \text{VIII}$$

From this formula, we may make several observations. If E is constant, as it will be because it is determined by the power rating of the transmitter, or the strength of the received signal; to make the current, I, as large as possible, which is what we want, Z must be made as small as possible. To make the wire resistance as small as possible, we use large wire sizes or tubing of low resistance conductors such as copper; to make X as small as possible, we *tune* the circuit (vary the inductance, L) to make $X_L=X_C$. This circuit and this formula are an excellent illustration of the use of mathematics to show *why* radio circuits are adjusted as they are and how mathematics is used not only to explain existing circuits to students but also to enable design engineers to improve circuits and develop new ones.

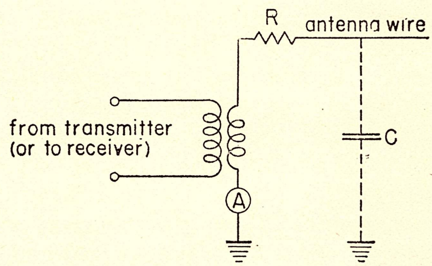


Fig. 31

PARALLEL CIRCUITS

The parallel circuit of R, L, and C is very commonly used in radio ransmitters and the radio-frequency sections of receivers because, as we shall see from formulas, it enables us to obtain *high* impedances in circuits when these high impedances are not obtainable in any other way. In Fig. 32,

R = wire plus circuit resistance
L = circuit inductance
C = circuit plus stray capacitance

From previous developments, we know that the impedance, Z, of the capacitive side of the parallel or *tank* circuit in complex-number form is $0-jX_C$ or $0-\frac{j}{2\pi fC}$. We also know that Z_2 is $R+jX_L$ or $R+j2\pi fL$. We know that the impedance of two parallel branches is $\frac{Z_1Z_2}{(Z_1+Z_2)}$. By the mathematical representation of parallel circuits:

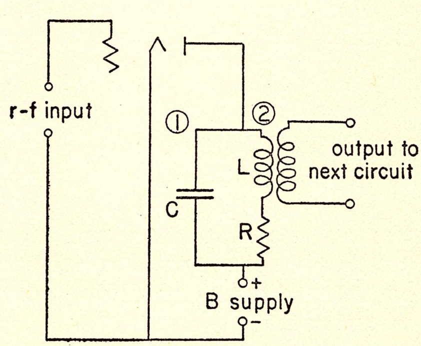


Fig. 32

$$Z=\frac{Z_1\,Z_2}{Z_1+Z_2}=\frac{\left(0-\frac{j}{2\pi fC}\right)(R+j2\pi fL)}{\left(0-\frac{j}{2\pi fC}\right)+(R+j2\pi fL)}=\frac{\left(-\frac{j}{\omega C}\right)(R+j\omega L)}{R+j\left(\omega L-\frac{1}{\omega C}\right)} \qquad \text{IXa}$$

By the algebra of complex numbers, we may multiply in the numerator and in the denominator to have:

$$Z=\frac{\frac{L}{C}-\frac{jR}{\omega C}}{R+j\left(\omega L-\frac{1}{\omega C}\right)}=\frac{\frac{L}{C}-\frac{jR}{\omega C}}{R+jX}\times\frac{R-jX}{R-jX}=\left(\frac{\frac{LR}{C}-\frac{RX}{\omega C}}{R^2+X^2}\right)-j\left(\frac{\frac{R^2}{\omega C}+\frac{XL}{C}}{R^2+X^2}\right). \qquad \text{IXb}$$

There are three possible conditions that may be stipulated for a parallel circuit to be *resonant.* The first is, let Z be maximum; the second is, let Z be real (*i.e.,* the *j term* is zero); the third is, let X be zero. It is found in practice that the three conditions of resonance occur almost simultaneously.

To find the first condition, the maximum value for Z, we use the process of differentiation which is part of the study of the calculus. Since the frequency is the principal or *independent* variable that determines Z, we differentiate Z with respect to f or to $2\pi f$, which is represented by omega (ω), and set the derivative equal to zero.

To find the second condition (*i.e.,* that Z is real), we set the *j term* equal to zero:

$$\frac{R^2}{\omega C}+\frac{XL}{C}=0=\frac{R^2}{\omega}+\left(\omega L-\frac{1}{\omega C}\right)L. \qquad \text{Xa}$$

Solving for ω, we find:

$$\omega=\sqrt{\frac{1}{LC}-\frac{R^2}{L^2}}. \qquad \text{Xb}$$

To find the third condition (*i.e.,* that X is zero), we set X_C equal to X_L. Solving for ω, we find:

$$\omega=\sqrt{\frac{1}{LC}}. \qquad \text{XI}$$

It is evident that, if $\frac{R^2}{L^2}$ is negligibly small, the conditions are identical, and that $Z=\frac{L}{CR}$. This, fortunately, is so in commercial equipment; and we use condition three in computations, condition one when we operate a transmitter (we tune for minimum plate current, which occurs when Z is maximum), and condition two to note that too much load (a large R) on an oscillator means unstable operation. This circuit and its analysis are another excellent illustration of the help mathematics is to understand *how* a circuit works and *how* to design it before building it in the laboratory or factory.

The circuit of Fig. 32 and the analytical work involved were based on the assumption that the resistance in the capacitive branch was zero. A very interesting observation may be made if it is assumed that Z_1 is not $0-jX_C$ but is R_C-jX_C, *i.e.,* there is resistance in the capacitive branch. As before, Z_2 is R_L+jX_L. Since branches one

and two are still in parallel, the basic equation for the impedance of parallel circuits still holds and:

$$Z=\frac{Z_1Z_2}{Z_1+Z_2}. \qquad \text{XII}$$

The algebra of multiplication and addition given above may be repeated, although the work is somewhat more laborious. To eliminate the j from the denominator, we previously multiplied numerator and denominator by $R-jX$; now we must multiply by $(R_L+R_C)-jX$. Collecting terms and expressing Z as a complex number, we have:

$$Z=\frac{R_CR_L(R_C+R_L)+R_C(\omega L)^2+R_L\left(\frac{1}{\omega C}\right)^2}{(R_L+R_C)^2+\left(\omega L-\frac{1}{\omega C}\right)^2}+j\frac{R_C^{\,2}\omega L-R_L^{\,2}\frac{1}{\omega C}-\frac{L}{C}\left(\omega L-\frac{1}{\omega C}\right)}{(R_L+R_C)^2+\left(\omega L-\frac{1}{\omega C}\right)^2}. \qquad \text{XIII}$$

If R_C is set equal to zero, and it is remembered that if $\omega L=\frac{1}{\omega C}$ we may write ω^2L^2 for $\frac{1}{\omega^2C^2}$ or $\frac{LR}{C}$ for $\frac{R}{\omega^2C^2}$; it will be found that the two equations for Z are identical. The second equation for Z is rather lengthy and involved but, if we make R_C equal to R_L and equal in magnitude to $\sqrt{\frac{L}{C}}$, substitution in the formula for Z will show the reactance of the circuit to be *zero* for all frequencies, since the numerator of the j term becomes zero. This substitution also shows that the resistance of the circuit becomes $\sqrt{\frac{L}{C}}$ and is *independent* of frequency. It is, perhaps, desirable that the algebra involved in this demonstration be given since it may not be obvious:

$$Z=\frac{\frac{L}{C}\left(2\sqrt{\frac{L}{C}}\right)+\sqrt{\frac{L}{C}}(\omega L)^2+\sqrt{\frac{L}{C}}\left(\frac{1}{\omega C}\right)^2}{4\frac{L}{C}+\left(\omega L-\frac{1}{\omega C}\right)^2}+j\frac{\frac{L}{C}\omega L-\frac{L}{C}\frac{1}{\omega C}-\frac{L}{C}\left(\omega L-\frac{1}{\omega C}\right)}{4\frac{L}{C}+\left(\omega L-\frac{1}{\omega C}\right)^2}$$

$$=\frac{\sqrt{\frac{L}{C}}\left[\frac{2L}{C}+\omega^2L^2+\frac{1}{\omega^2C^2}\right]}{4\frac{L}{C}+\omega^2L^2-2\frac{L}{C}+\frac{1}{\omega^2C^2}}+j\frac{\frac{L}{C}\left(\omega L-\frac{1}{\omega C}\right)-\frac{L}{C}\left(\omega L-\frac{1}{\omega C}\right)}{\omega^2L^2+2\frac{L}{C}+\frac{1}{\omega^2C^2}}=\sqrt{\frac{L}{C}}+j0. \qquad \text{XIV}$$

This observation, unfortunately, cannot usually be used commercially since C involves stray capacity in a circuit as well as a physical condenser and R_C therefore cannot be employed as the equation shows. However, it does show that "tank" circuits, if "loaded" by resistors, may be used for wide frequency bands. This idea has been useful in designing television video amplifiers.

ANTENNA CIRCUIT

Now that both series and parallel resonant circuits have been considered, we may return to the antenna of Fig. 31 and introduce a few more circuit elements, as shown in Fig. 33. Such an antenna circuit is sometimes used to suppress some particular signal frequency, and L_1C_1 is called a "wave trap", while other frequencies are received. If L_1C_1 is tuned to the undesired frequency the impedance, Z, of L_1C_1 to that frequency is resistive and *large*. Therefore the current flow in the antenna circuit at that frequency will be *small*; therefore IX_{L_2} at that frequency will be small and the undesired signal will be effectively suppressed. However, at other frequencies the impedance of L_1C_1 will be $R+jX$ where R is relatively small and the reactance, X, may be "tuned out" by adjusting C_2 so that X, C_A, C_2, and L_2 form a series resonant circuit. Then the only impedance in the antenna circuit for the *desired* frequency will be R_A plus R. The current will be relatively large and therefore the desired signal voltage across L_2 relatively large.

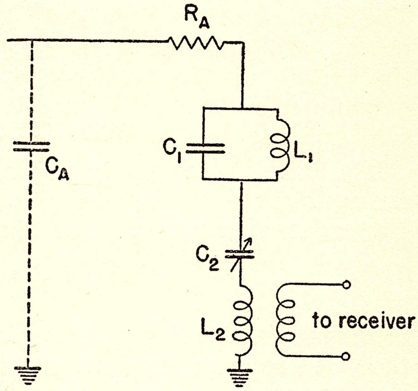


Fig. 33

Properties of vacuum tubes

Since radio circuits are designed around vacuum tubes, the electrical properties of vacuum tubes are all-important. These properties are conventionally expressed in terms of three "constants".

PLATE RESISTANCE

The first is the *plate resistance*, r_p, of the tube, which is the ratio of the change in plate voltage causing a change in plate current. In algebraic notation, the statement is:

$$r_p = \frac{E_{p_2} - E_{p_1}}{I_{p_2} - I_{p_1}} = \frac{\Delta E_p}{\Delta I_p}. \qquad \text{XV}$$

where ΔE_p (delta E sub p) = the change in plate voltage,

ΔI_p (delta I sub p) = the change in plate current.

In calculus notation, the delta E_p, which indicates a finite change in E_p is replaced by ∂E_p, which indicates an infinitesimally small change in E_p. Then:

$$r_p = \frac{\partial E_p}{\partial I_p}. \qquad \text{XVI}$$

AMPLIFICATION FACTOR

The *amplification factor*, μ, which denotes the signal amplification which a tube may effect, is the ratio of the *increase* in *plate* voltage needed to keep the plate current constant if the control *grid* voltage is *decreased*. In mathematical notation, the statement is:

$$\mu=\frac{E_{p_2}-E_{p_1}}{E_{g_2}-E_{g_1}}=\frac{\Delta E_p}{\Delta E_g}=\frac{\partial E_p}{\partial E_g}. \qquad \text{XVII}$$

TRANSCONDUCTANCE

The third of these properties is the *transconductance*, S_m, and is the ratio of the amplification factor of the tube to the plate resistance. The mathematical statement is:

$$S_m=\frac{\mu}{r_p}=\frac{\partial E_p}{\partial E_g}\div\frac{\partial E_p}{\partial I_p}=\frac{\partial I_p}{\partial E_g}. \qquad \text{XVIII}$$

It may be noticed from this formula that transconductance involves the ratio of current in the plate circuit to voltage in the grid circuit. Two observations should be made: first, that the fraction is the inverse of the usual Ohm's law form; and second, that the voltage is in the input circuit of the vacuum tube while the current is in the output circuit of the tube. Thus, the mathematical formula gives us a means to see the effect of one electrical circuit on another. The formula

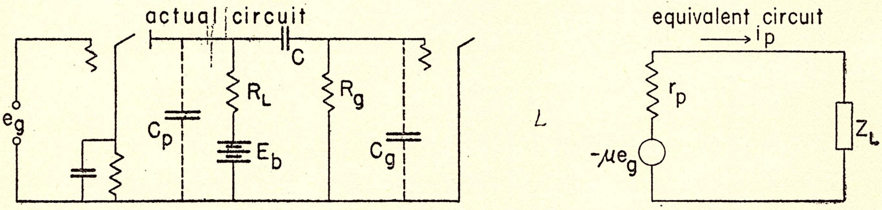


Fig. 34

to express the actual gain of an audio amplifier shows the application of the above *tube constants*.

By Ohm's law:

$$i_p=\frac{-\mu e_g}{r_p+Z_L}$$

and the voltage across the load equals

$$i_pZ_L=\frac{-\mu e_gZ_L}{r_p+Z_L}. \qquad \text{XIXa}$$

By dividing numerators and denominators by r_p, we get

$$i_p=\frac{-\mu}{r_p}\times\frac{e_g}{1+\frac{Z_L}{r_p}}=-S_me_g\frac{r_p}{r_p+Z_L}$$

$$i_pZ_L=-S_me_g\frac{r_pZ_L}{r_p+Z_L}. \qquad \text{XIXb}$$

This equation shows that the voltage impressed on the grid of the second stage is that voltage, e_g, impressed on the grid of the stage being studied multiplied by the transconductance of the tube times a "percentage factor". The value of this factor obviously depends on the relative values of the tube's plate resistance, r_p, and the load impedance, Z_L. The minus sign indicates that the voltage is *inverted*—that is, reversed 180° in phase.

The next consideration in this problem is the nature of Z_L. In order to simplify computations, the circuit parameters, C_p, R_L, C, R_g, and C_g, considered should be kept as few in number as possible. Thus, if we wish to consider them *all*, we have

$$Z_L = \cfrac{1}{\cfrac{1}{-jX_{C_p}} + \cfrac{1}{R_L} + \cfrac{1}{\cfrac{-jR_gX_{C_g}}{R_g - jX_{C_g}} - jX_C}}. \qquad \text{XXa}$$

Obviously, this expression is rather cumbersome; if some of the quantities may be neglected, the work will be much simplified. At low frequencies, we may neglect C_p and C_g and the expression for Z reduces to:

$$Z = \cfrac{1}{\cfrac{1}{R_L} + \cfrac{1}{R_g - jX_C}}. \qquad \text{XXb}$$

At intermediate frequencies, we may *still* neglect C_p and C_g, and also neglect the impedance of C. The expression for Z now reduces to:

$$Z = \cfrac{1}{\cfrac{1}{R_L} + \cfrac{1}{R_g}} = \frac{R_L R_g}{R_L + R_g}. \qquad \text{XXc}$$

At high frequencies, we may no longer neglect C_p and C_g, but we may neglect the impedance of C. The expression for Z now becomes:

$$Z = \cfrac{1}{\cfrac{1}{R_L} + \cfrac{1}{R_g} + \cfrac{1}{-jX}}, \qquad \text{XXd}$$

where X is the reactance of C_p and C_g in parallel.

It is evident from these analyses that, in order to simplify the mathematical work involved in any particular problem, circuit-simplifying assumptions may be made. While the design engineer is fully aware of the fact that so doing introduces errors into his work, if these errors are negligibly small, the diminution in the mathematical work involved justifies the simplifications made.

GEOMETRY IN RADIO

The illustrations of the applications of mathematics to radio thus far given are primarily algebraic. Geometry also is valuable, although not so constantly and obviously used. The right triangle has already been mentioned in conjunction with the complex number. Analytic geometry shows that, if a source of light (or radio energy) is placed at the focal point of a parabola, the rays of light (or energy) reflected will be parallel as illustrated in Fig. 35. The knowledge is used to direct the radiation from an antenna, or the light from a lamp in a motion-picture projector, or to concentrate the sound pick-up by a microphone in an auditorium.

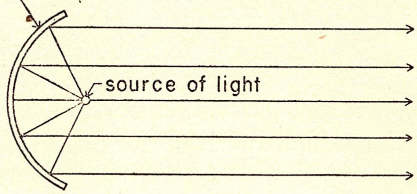


Fig. 35

Variation of impedance

A geometric graph (or "picture") of the variation of the impedance of a coil with frequency is *linear*, because the graph of the equation,

$$X_L = (2\pi L)f,$$

is a straight line. This is obvious when we note that X and f both have the exponent, 1. However, a geometric graph of the variation

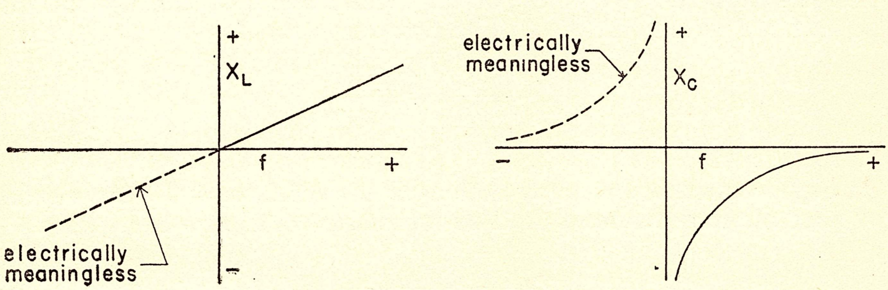


Fig. 36

of the impedance of a capacitor with frequency is a hyperbola because that is the graph of the equation,

$$X_C = \frac{1}{(2\pi C)f} \text{ or } X_C f = \frac{1}{2\pi C} = k \text{ (a constant).}$$

This is obvious when we note that the equation is quadratic in form.

Operation of vacuum tube

In the analysis of the operation of a vacuum tube, we frequently assume the *characteristic curve* to be that of the parabola since the equation of the parabola is quadratic and we know how to handle quadratic equations without too much trouble.

Fig. 37 shows in a solid line the true curve, while the dashed lines show what a parabola would be. If we confine our study of the tube to the portion where the two curves coincide, our equations will be quite correct.

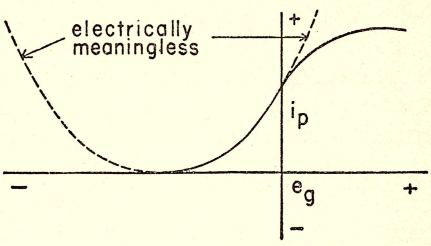


Fig. 37

Sound volume

The graph of logarithms is characteristic and shows that the logarithm of 1 is 0, and the logarithm of any number greater than 1 is greater than 0, while the logarithm of any number between 0 and 1 is negative.

The sensation of sound volume is not directly proportional to the sound volume causing that sensation. The sound of two sources activated simultaneously is not twice as loud as that of one, but approximately 1.3 times as loud. This, we discover, follows the laws of logarithms. Similarly, the power obtainable at the end of an electrical-transmission line is related to the power inserted at the sending end by the laws of logarithms. The law is stated as follows: the logarithm to the base 10 of the ratio of the power in to the power out shall be designated as one bel. If we use the *decibel* (one-tenth of a bel), which is the commonly used unit,

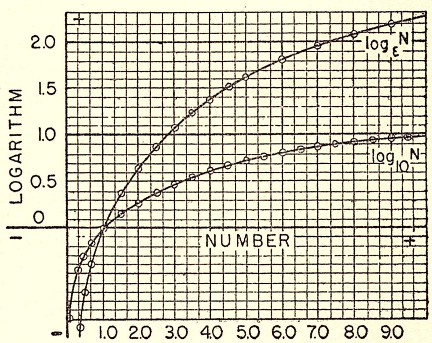


Fig. 38

$$\text{the number of decibels, } db = 10 \log_{10} \frac{P_{in}}{P_{out}}.$$

TRIGONOMETRY IN RADIO

Trigonometry is a branch of the subject of mathematics that is absolutely vital in the study of radio, since all waves of electrical or magnetic energy may be analyzed as composed of one or many sine waves. In radio, most waves are other than sinusoidal in form, but it can be shown by Fourier Analysis (usually studied with the calculus) that a complex wave may be broken down into a sine wave plus harmonics*.

* Harmonics are sine waves with frequencies which are whole-number multiples of the fundamental frequency.

The equation of such a wave may be written as follows:

$$e = E_1 \sin \omega t + E_2 \sin 2\omega t + E_3 \sin 3\omega t + \ldots \qquad \textbf{XXIa}$$

or, if it is necessary to show time displacement of the harmonics, we show it in this manner:

$$e = E_1 \sin \omega t + E_2 \sin (2\omega t + \theta_2) + E_3 \sin (3\omega t + \theta_3) + \ldots \qquad \textbf{XXIb}$$

In Fig. 37, we made the observation that the characteristic curve of a vacuum tube is at least parabolic, certainly non-linear. If we again assume a square-law (parabolic) characteristic curve, and the voltage on the grid is

$$e = E \sin \omega t$$

and the plate current is given by the equation,

$$I_p = [E_p + \mu\, e]^2,$$

then, by substituting the first equation in the second, we have:

$$I_p = [E_p + \mu\, E \sin \omega t]^2. \qquad \textbf{XXIIa}$$

By the algebra of squaring a binomial, we find:

$$I_p = E_p^2 + 2\mu\, E_p\, E \sin \omega t + \mu^2\, E^2 \sin^2 \omega t. \qquad \textbf{XXIIb}$$

By the laws of trigonometry, we find:

$$I_p = E_p^2 + 2\mu\, E_p\, E \sin \omega t + \frac{1}{2}\mu\, E^2 - \frac{1}{2}\mu^2\, E^2 \cos 2\omega t. \qquad \textbf{XXIIc}$$

The first of the four terms on the right-hand side of this equation is constant and tells us that direct current flows; the second is sinusoidal, as is the signal on the grid, and is the desired and useful term; the third is constant like the first; while the fourth shows the presence of an unwanted second harmonic, since the number, 2, appears in front of the omega.

Amplitude modulation

Another readily shown application of trigonometry to radio is the demonstration of the process of *amplitude modulation.* The justification for the use of the equations in the previous illustration and this one is involved and requires considerable discussion and analysis. It is sufficient now to say that the work can be amply justified.

In an antenna, the current (and therefore the radiated energy) may be shown to be in accordance with the equation,

$$i = I \sin 2\pi f_c t \qquad \textbf{XXIIIa}$$

This equation is true if the signal is unmodulated (*i.e.*, is a pure radio-frequency sine wave with no audio signal impressed upon it). If the audio signal, $m\, I \sin 2\pi f_a t$, is impressed upon it, the equation becomes:

$$i = (I + m\, I \sin 2\pi f_a t) \sin 2\pi f_c t, \qquad \textbf{XXIIIb}$$

where f_c = the transmitter station carrier frequency, such as 660 Kc.
f_a = the audio modulating frequency,
i = the instantaneous current in the antenna,
I = the peak value of the carrier current,
m = the per cent of modulation.

By algebra, multiplying a binomial by a monomial:

$$i=I \sin 2\pi f_c t+m\, I \sin 2\pi f_c t \sin 2\pi f_a t. \qquad \text{XXIIIc}$$

By trigonometry:

$$i=I \sin 2\pi f_c t-\frac{1}{2}\, m\, I \cos 2\pi\, (f_c+f_a)\, t+\frac{1}{2}\, m\, I \cos 2\pi\, (f_c-f_a)\, t. \qquad \text{XXIIId}$$

It is evident that, by amplitude-modulating a *single* carrier frequency by a *single* audio frequency, we obtain *three* frequencies as a result: the first is the carrier as it existed before modulation was applied; the second is a higher frequency which lies in what is called the *upper* side band; the third is a lower frequency which lies in what is called the *lower* side band. It is this analysis which shows why radio stations must be spaced in the frequency spectrum to avoid interference with one another.

Frequency modulation

The mathematical analysis of frequency modulation, commonly known and used today, is similar but much more involved. As in the case of amplitude modulation, we may begin with the unmodulated wave, as:

$$i=I \sin \omega_c t=I \sin 2\pi f_c t.$$

The term, *frequency modulation*, means that the frequency, f_C, is varied by the modulating signal,

$$mf \sin 2\pi f_a t=mf \sin \omega_a t.$$

The equation for i then becomes:

$$i=I \sin (\omega_c t+m\, \omega_c t \sin \omega_a t). \qquad \text{XXIVa}$$

By trigonometry, letting $m\, \omega_C t$ be represented by n:

$$i=I[\sin \omega_c t \cos (n \sin \omega_a t)+ \cos \omega_c t \sin (n \sin \omega_a t)\,]. \qquad \text{XXIVb}$$

This equation is much more difficult to interpret than the equation obtained for amplitude modulation, since it involves a trigonometric function of a trigonometric function which has no significance to us as it stands. In order to obtain terms in the expression which have electrical meaning to us, we must resort to the mathematical facts that:

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ \ldots$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ \ldots$$

If we let x in these formulas be replaced by $n \sin \omega_a t$, it becomes evident that i is the sum of *two* infinite series involving trigonometric functions raised to increasingly higher powers. (The work involved in doing this substitution is not difficult but is time-consuming

and is left to the reader for practice work.) Since trigonometric functions raised to powers have no obvious significance in the explanation of frequency modulation (a similar condition was observed before in the discussion of the square-law amplifier), it is necessary that we again resort to well-known formulas of trigonometry:

$$\sin^2 x=\frac{1}{2}-\frac{1}{2}\cos 2x$$

$$\sin^3 x=\frac{3}{4}-\frac{1}{4}\sin 3x$$

$$\sin^4 x=\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$$

$$\sin^5 x=\frac{5}{8}\sin x-\frac{5}{16}\sin 3x+\frac{1}{16}\sin 5x,\text{ etc.}$$

By substituting these equations in the terms of the infinite series, we may resolve i into an equation of the same *form* as that obtained for amplitude modulation:

$$\begin{aligned} i=I\Bigg[&\left(1-\frac{1}{2}\cdot\frac{n^2}{2!}+\frac{3}{8}\cdot\frac{n^4}{4!}+\ .\ .\ .\right)\sin\omega_c t\\ &+\left(n-\frac{3}{4}\cdot\frac{n^3}{3!}+\frac{5}{8}\cdot\frac{n^5}{5!}+\ .\ .\ .\right)\cos\omega_c t\sin\omega_a t\\ &+\left(\frac{1}{2}\cdot\frac{n^2}{2!}-\frac{1}{2}\cdot\frac{n^4}{4!}+\frac{15}{32}\cdot\frac{n^6}{6!}+\ .\ .\ .\right)\sin\omega_c t\cos 2\omega_a t\\ &+\left(\frac{1}{4}\cdot\frac{n^3}{3!}-\frac{5}{16}\cdot\frac{n^5}{5!}+\frac{21}{64}\cdot\frac{n^7}{7!}+\ .\ .\ .\right)\cos\omega_c t\sin 3\omega_a t+\ .\ .\ .\Bigg].\end{aligned} \qquad \text{XXIVc}$$

This equation shows that frequency modulation produces the carrier, just as in amplitude modulation except that its *amplitude* depends on the modulation, plus *an infinity of side bands*. However, these side bands diminish in amplitude and, if we make the channel assigned to each F. M. transmitter wide enough, the side bands beyond this range are too small to create interference with adjacent channel stations and may therefore be ignored. The coefficients written above are infinite series and their evaluation is very laborious. However, they appear in other mathematical works and are called Bessel's functions, conventionally designated by J's with subscripts corresponding to the power of n in the first term of each series. This notation does not simplify the problem any, but does permit a briefer equation:

$$\begin{aligned} i=I\ [\ &J_0(n)\sin\omega_c t+2\ J_1(n)\cos\omega_c t\sin\omega_a t\\ &+2\ J_2(n)\sin\omega_c t\cos 2\omega_a t+2\ J_3(n)\cos\omega_c t\sin 3\omega_a t+\ .\ .\ .] \end{aligned} \qquad \text{XXIVd}$$

The *products* of trigonometric functions shown may be replaced as in amplitude-modulation analysis by sum-and-difference terms to yield

the final equation showing carrier and side bands, and reads as follows:

$$i = I[J_0(n) \sin \omega_c t + J_1(n) \sin 2\pi (f_c+f_a)t - J_1(n) \sin 2\pi (f_c-f_a)t$$
$$- J_2(n) \sin 2\pi (f_c+2f_a)t + J_2(n) \sin 2\pi (f_c-2f_a)t$$
$$+ J_3(n) \sin 2\pi (f_c+3f_a)t - J_3(n) \sin 2\pi (f_c-3f_a)t + \ldots]. \quad \textbf{XXIVe}$$

Neutralization

Another application of mathematics to radio is the explanation of *neutralization*. When triode tubes are used in transmitters to amplify the carrier-frequency signal, it is almost invariably necessary to *neutralize* the stages to prevent feedback (and therefore oscillation). A common method is to set up the circuit in the form of a Wheatstone bridge (Fig. 39), and to balance the bridge by tuning C_N with no plate

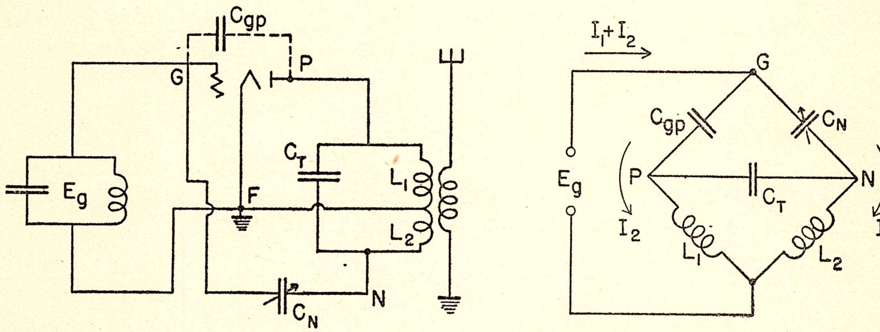


Fig. 39

voltage on the stage so that there is no output (*i.e.*, the voltage between the points, P and N, is zero). By the laws of the Wheatstone bridge:

$$-j\, I_1\, X_{CN} = -j\, I_2 X_{C_{gp}}$$
$$j\, I_1\, X_{L_2} = j\, I_2\, X_{L_1}.$$

By the laws of algebra, and remembering that $X_L = 2\pi f L$ and $X_C = \dfrac{1}{2\pi f C}$, we have:

$$C_N = C_{gp} \times \frac{L_1}{L_2} \qquad \textbf{XXV}$$

This equation gives us a ready means of deciding where to adjust the tap on the coil (*i.e.*, determine the ratio of turns on L_1 to the turns on L_2) and to determine the size (capacity) of the neutralizing condenser since the stray capacity, C_{gp}, of the tube is given by the tube manufacturer in a tube manual.

THE CALCULUS IN RADIO

To show how to adjust an oscillator involves the use of the calculus. An oscillator, of course, is the vacuum tube and circuit that generates the high frequency sine waves previously called "the carrier frequency" when modulation was discussed. The triode vacuum tube exhibits the property of *negative resistance.* To show this, we may develop the circuit parameters of a conventional electrical circuit that will oscillate, if its parameters are physically possible. Fig. 40a shows the

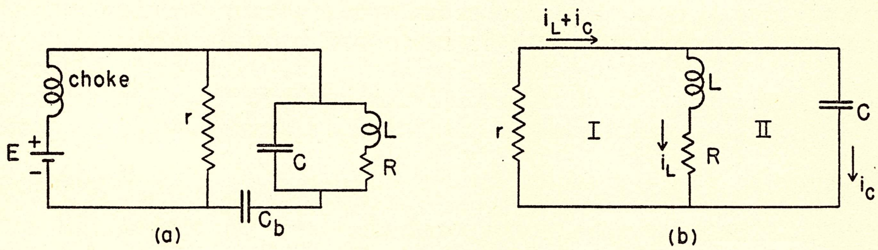


Fig. 40

circuit, and Fig. 40b the A.-C. part in which we are interested. By Kirchhoff's law, which says that the sum of *all* the voltages in a closed circuit is zero, we may say for circuit I that:

$$r(i_L+i_C)+i_LR+Lpi_L=0. \qquad \textbf{XXVI}$$

The symbol, p, represents the differential calculus operator, $\frac{d}{dt}$. This substitution of symbols is commonly employed because it makes writing, typing, and printing equations much easier. This equation involves the *two* currents, i_L and i_C. We must, therefore, find another equation involving them if we are to solve the problem. Circuit II supplies the second equation, since we may again apply Kirchhoff's law to obtain the equation,

$$Lpi_L+i_LR-\frac{1}{C}\int i_Cdt=0 \qquad \textbf{XXVIIa}$$

Since we do not know how to handle this equation in this form, we may differentiate both sides to obtain it in the form:

$$Lp^2i_L+Rpi_L-\frac{1}{C}i_C=0 \qquad \textbf{XXVIIb}$$

By algebra, we may solve the equation for circuit I for i_C and substitute this value in the equation for circuit II. If this is done, we

obtain a *single* equation involving only *one* unknown, i_L. This process was used in the solution of simultaneous equations in elementary algebra.

$$p^2 i_L + \left(\frac{R}{L} + \frac{1}{Cr}\right) p i_L + \frac{1}{L\,C}\left(\frac{R}{r} + 1\right) i_L + 0.$$

This equation involves a derivative of the second order in the first term, a derivative of the first order in the second term, and the variable, i_L to the first power, in the third term. Differential equations show us that, if i_L is to be *sinusoidal* (which means if the oscillator is to function), the coefficient of the pi_L term must be zero.

$$\therefore \quad \frac{R}{L} + \frac{1}{Cr} = 0 \quad \text{and} \quad r = -\frac{L}{CR}.$$

It is well known (and noted before in this article) that the impedance of a tank, or parallel resonant circuit, is $\frac{L}{CR}$ if R is small. Evidently the circuit parameter, r, must equal the tank impedance and be *negative* in sign. The vacuum tube is the most commonly-known piece of electrical apparatus that can exhibit the property of negative resistance, and it is therefore used as a sine-wave generator for radio frequencies. When the oscillator takes the form of Fig. 41, which

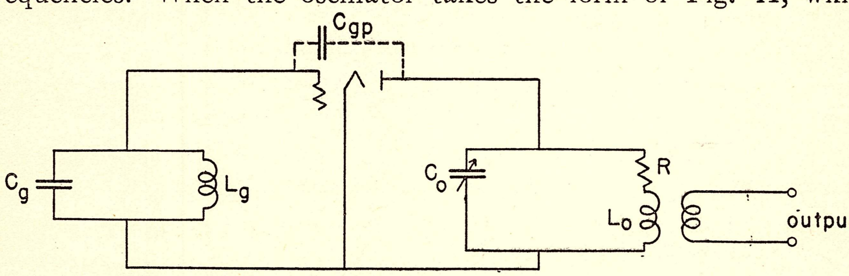


Fig. 41

is known as a tuned-grid tuned-plate oscillator, another important mathematical analysis occurs. By writing Kirchhoff equations as we did before, we find it possible to express the impedance which the vacuum tube presents to the grid tank circuit, C_gL_g. This impedance has a real part, R_g, as follows (the imaginary part is here immaterial):

$$R_g = \frac{r\,[\,A + B - \mu X_o X\,]}{D} \qquad \text{XXVIII}$$

where r = the plate resistance of the tube
A = always positive and determined by r and R
B = always positive and determined by μ and $L_o\,C_o$
D = always positive and determined by r, R, and $L_o\,C_o$
X = always positive and determined by the value of C_{gp}
X_o = positive or negative, depending on the frequency to which L_oC_o is tuned.

Since we know that R_g must be negative and *just* as large as the positive resistance of the grid tank circuit, our problem is how to adjust the output tank circuit to effect this condition. From the

mathematical equation, X_o must be *positive* and sufficiently large to make R_g of the proper magnitude. This means that the plate-tank circuit must be tuned to a *higher* frequency than the grid-tank circuit (then X_o is positive) and enough higher so that R_g is large enough. This condition is observed experimentally by the flow of grid current.

Decrement

An illustration of a similar use of the calculus is the discussion of *decrement*. This analysis formerly was used primarily in the explanation of the operation of the spark transmitter. However, the spark transmitter's chief fault, the decrement of the tank circuit, is present in all tank circuits. In circuit II of Fig. 40b, if this circuit stands alone, the differential equation is:

$$Lpi+iR+\frac{1}{pC}=0 \qquad \text{XXIX}$$

Differential calculus shows that there are three possible solutions:

$$\textbf{a}\quad i=-\frac{E}{\beta L}\,\varepsilon^{-\alpha t}\sinh\beta t$$

$$\textbf{b}\quad i=-\frac{Et}{L}\,\varepsilon^{-\alpha t}$$

$$\textbf{c}\quad i=-\frac{E}{\omega L}\,\varepsilon^{-\alpha t}\sin\omega t$$

where $\alpha=\dfrac{R}{2L}$

$$\beta=\sqrt{\alpha^2-\frac{1}{LC}}$$

sinh = hyperbolic sine
sin = circular (trigonometric) sine

Only the third solution is of interest to us, since this is the only solution with a sine function, which means oscillations, which, of course, is what we need. This condition obtains when R is *small*. If we replace omega (ω) by $\dfrac{1}{\sqrt{LC}}$, the third solution becomes:

$$i=\left(-E\sqrt{\frac{C}{L}}\right)\varepsilon^{-\frac{Rt}{2L}}\left(\sin\frac{t}{\sqrt{LC}}\right) \qquad \text{XXX}$$

The first factor in this solution shows the maximum amplitude that the current may have and clearly shows that the voltage impressed on the circuit is all-important. The second factor becomes smaller as time increases and shows the attenuation of the current with successive oscillations of the current. The third factor shows that the current is oscillatory and that the period (or frequency) of oscillations depends on the values of L and C. It is the second factor in which our interest now centers; it shows how rapidly the oscillations diminish with time t and the exponent, $\dfrac{Rt}{2L}$, is called "the decrement" of the

circuit. Since the period of a cycle is the reciprocal of the frequency, we may write the decrement, δ, in the more usual form,

$$\delta=\frac{R}{2fL}. \qquad \text{XXXI}$$

The decrement of a tank circuit is less than 1, and may be 0.2, as it was in the spark transmitters.

In the case of the old spark radiator, the oscillations occurred at an r-f frequency of, say, 500 kilocycles and some 24 useful oscillation cycles occurred. The 25th oscillation was too feeble to be considered; that is, the tank voltage was too small by that time to give appreciable radiation. The energy replacement necessary for radiation occurred at an audio frequency of, say, a 1000-cycle rate. Obviously, the energy replacement occurred at a far slower rate than the energy dissipation. The resulting signal was called a "damped wave".

Part of the study of *decrement* is the determination of the time when the current crest occurs. To find this, differentiation enters the problem. If we differentiate the equation for i with respect to t and set the result equal to 0, we may determine the value:

$$\frac{di}{dt}=pi=-E\sqrt{\frac{C}{L}}\,\varepsilon^{-\frac{Rt}{2L}}\left(\cos\frac{t}{\sqrt{LC}}\right)\frac{1}{\sqrt{LC}}$$

$$-E\sqrt{\frac{C}{L}}\left(\sin\frac{t}{\sqrt{LC}}\right)\varepsilon^{-\frac{Rt}{2L}}\left(-\frac{R}{2L}\right)=0$$

By algebra and trigonometry:

$$\frac{R\sqrt{C}}{2\sqrt{L}}\sin\frac{t}{\sqrt{LC}}=\cos\frac{t}{\sqrt{LC}}=\sqrt{1-\sin^2\frac{t}{\sqrt{LC}}}$$

Squaring both sides:

$$\frac{R^2C}{4L}\sin^2\frac{t}{\sqrt{LC}}=1-\sin^2\frac{t}{\sqrt{LC}}$$

$$\left(\frac{R^2C}{4L}+1\right)\sin^2\frac{t}{\sqrt{LC}}=1$$

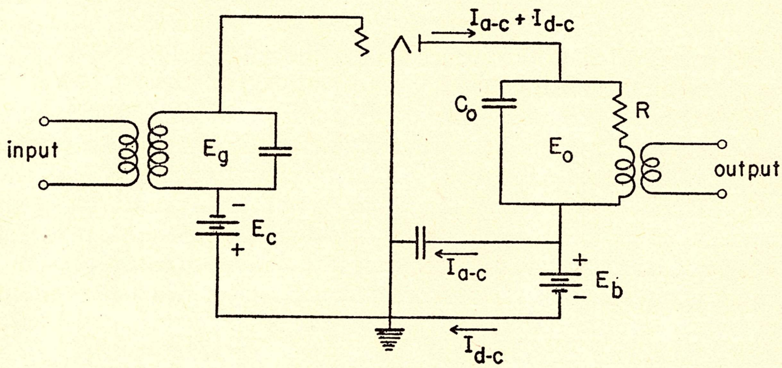
$$\sin^2\frac{t}{\sqrt{LC}}=\frac{4L}{R^2C+4L}$$

$$t=\sqrt{LC}\,\sin^{-1}\sqrt{\frac{4L}{R^2C+4L}} \qquad \text{XXXII}$$

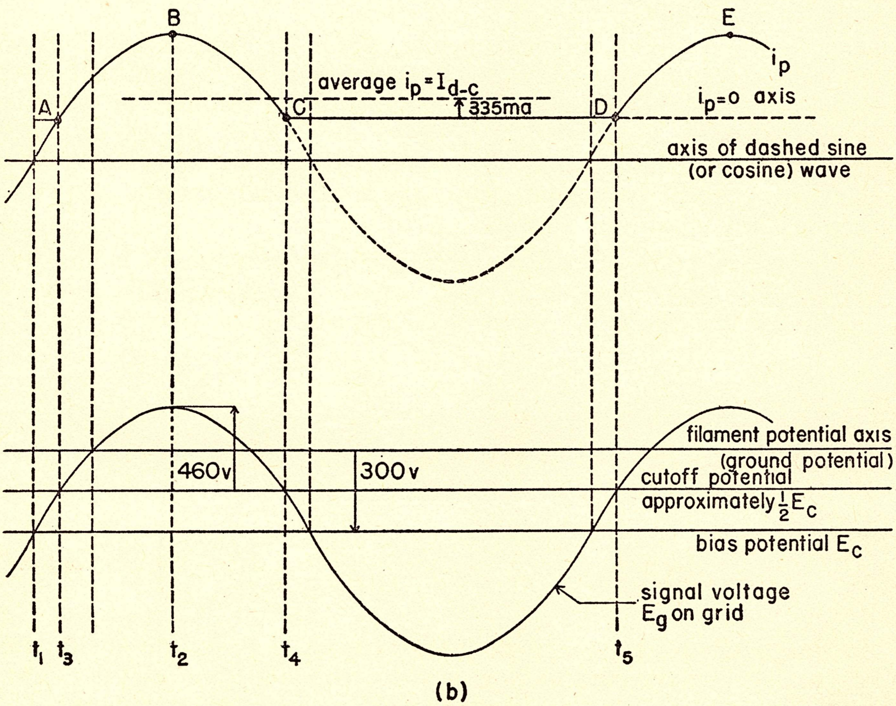
Amplifier stages

The design of amplifier stages for transmitters is an extremely important and rather difficult problem for the radio-design engineer. It is important that the stages be designed to operate at the highest

possible efficiency to keep the size of the equipment as small as possible and the cost of the transmitter and also its operation within



(a)



(b)

Fig. 42

reasonable bounds. As usual, mathematics is one of the important mental tools the design engineer requires, but the analytical work

is difficult because of the form of the current and voltage pulses that are encountered, as Fig. 42 shows. From Fig. 42, it is apparent that the grid has a negative D.-C. potential, E_C, impressed upon it. A typical case is the 833 tube, used in many modern broadcast-station transmitters, in which a typical 833 stage would have E_b set at 2500 volts, E_g at 460 volts, E_C at minus 300 volts, the D.-C. plate current at 335 milliamperes (0.335 amperes), and the r-f power output at 635 watts.

Impressed on the bias potential as an axis is a "fluctuating" signal voltage. If the analysis of the graph in Fig. 42 is started at time t_1, this fluctuating signal voltage is a sine wave; if it is started at time t_2, the voltage is a cosine wave. Thus far in our analysis, which starting time is chosen is immaterial since the mathematical operations with sine and cosine curves are about of the same nature and difficulty.

However, our major interest is in the plate current, i_p, curve *ABCDE*, and an equation for it. This curve, obviously, is *not* a simple sine or cosine curve or any other simple mathematical curve. If we try to start our analysis at the time, t_1, the analytical problem becomes extremely difficult but, since we know that the plate current is zero between the point, C (time t_4), and the point, D (time t_5), we do not need to worry about any current equation for that interval. If we start the study at point B (time t_2), we know that the curve is cosinusoidal between the points, B and C, and, if we confine our study to that region, the curve may be so treated. In fact, we are primarily concerned with the location of the point, C, and we know that, when the plate current is at that point, the D.-C. voltages and the A.-C. voltages are equal.

The justification of this statement is the province of radio textbooks. The equation that may be written, with the A.-C. voltages on the left and the D.-C. voltages on the right is:

$$\left(E_g-\frac{E_p}{\mu}\right)\cos\omega\,(t_4-t_2)=-\left(E_C+\frac{E_b}{\mu}\right). \qquad \text{XXXIII}$$

Since we are interested in the value of $\omega(t_4-t_2)$, which is called *the angle of plate current flow*, we solve for it, and obtain the equation:

$$\cos\omega\,(t_4-t_2)=\left[-\left(E_C+\frac{E_b}{\mu}\right)\right]\div\left[E_g-\frac{E_p}{\mu}\right]. \qquad \text{XXXIV}$$

As this angle covers the time interval from t_2 to t_4, the entire time (or angle) of plate current flow is twice the angle, since it is evident from Fig. 42 that the plate current flows from time t_3 to time t_4. This problem is a good illustration of the fact that the radio engineer frequently uses equations to represent a *part* of a problem and must keep in mind the fact that his mathematical work must constantly

be guided by the electrical features of the problem and mathematical results must be interpreted in terms of the electrical facts.

Tank circuits

In the design of tank circuits for transmitters, it is necessary to select some values for the inductor and the capacitor. Thc equation relating these two is

$$f=\frac{1}{2\pi\sqrt{LC}},$$

where f is the frequency and is, of course, known for any given transmitter; the frequency is assigned by the Federal Communications Commission. However, we have two unknowns, L and C, and but one equation. It is desirable that a reasonable value be selected for C so that L may be computed. This is done by assuming that:

a in Fig. 42, plate current flows 50% of the time—*i.e.*, that $\omega(t_4-t_2)$ is 90°;

b the maximum A.-C. voltage, E_o, across the tank circuit in Fig. 42 is equal to E_b;

c we remember from trigonometry that the *average* of a sine (or cosine) wave is 0.636 of the maximum value;

d we know that $E_o = I_{\text{A.C.}}Z$, where Z is the tank impedance at parallel resonance, and is equal to $\frac{L}{CR}$, which is equal to $\frac{\omega^2L^2}{R}$;

e we know that "the figure of merit" of a tank circuit is Q, and is equal to $\frac{\omega L}{R}$ and in an operating circuit has a value between 10 and 15.

With these assumptions, we may write the equations:

$$E_o=I_{\text{A.C.}}Z=E_b=\frac{I_{\text{D.C.}}}{0.636}Z=1.57\,I_{\text{D.C.}}Z$$

$$E_b=\frac{1.57Q}{\omega C}\,I_{\text{D.C.}} \qquad \textbf{XXXVa}$$

Therefore, by algebra:

$$C=\frac{1.57QI_{\text{D.C.}}}{\omega\,E_b} \qquad \textbf{XXXVb}$$

Since Q is generally known, $I_{\text{D.C.}}$ and E_b are given by the tube manufacturer, and ω is given by the F.C.C., we have all the data to compute C. We then go to the earlier equation and compute L.

TELEVISION AND HIGH-FREQUENCY TRANSMISSION

Television and other high-frequency transmitters involve a simple problem in geometry and algebra. This is the transmission distance possible assuming that the radiated signals travel in straight lines and that the transmission distance is thereby limited to "line of sight". Fig. 43 shows the problem.

The points, T and R, represent respectively the transmitting and receiving antennas. The symbol, r, is the radius of the earth, and h is the height of the transmitting antenna. From geometry, the angle, TRC, is a right angle; from the Pythagorean theorem:

$$r^2+m^2=(r+h)^2=r^2+2rh+h^2.$$

Since h is very small as compared to r, and m and d are practically identical, we may say

$$m^2=d^2=2rh.$$

Assuming r to be 4000 miles,

$$d\ (\text{miles})=1.23\sqrt{h\ (\text{feet})}$$

Experience has shown that this equation is quite reasonable, with occasional exceptions in which satisfactory reception has been discovered at greater distances.

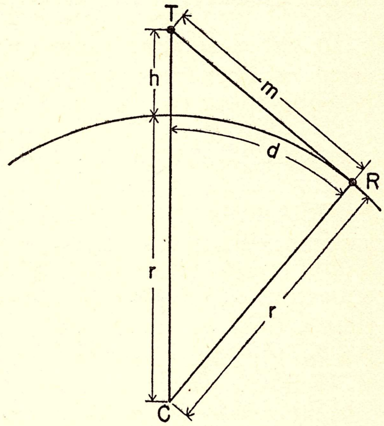


Fig. 43

Separating signals

There are two circuits used to separate signals in television receivers. The explanation of their action involves the use of differential equations. If it is desired to separate widely-separated radio frequencies,

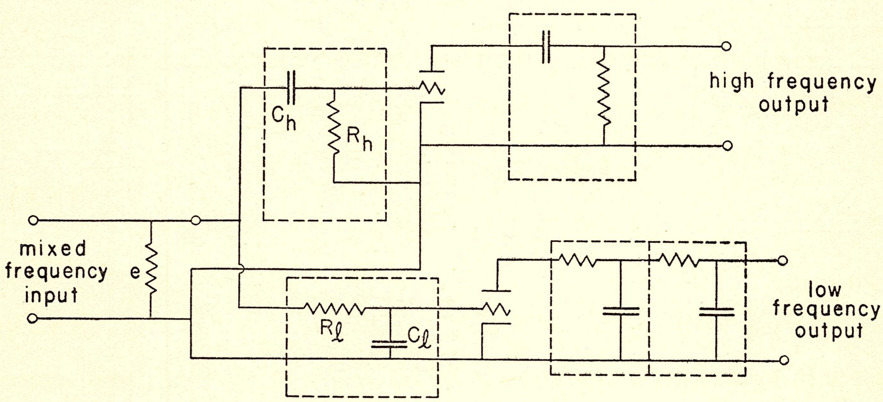


Fig. 44

we usually employ tuned tank circuits; but in this case we wish to separate the vertical and the horizontal synchronizing frequencies which are radiated along with the television-picture frequencies. The synchronizing frequencies are needed to keep the electron beam in the receiver kinescope sweeping up and down as well as back and forth exactly in synchronism with the electron beam in the television camera iconoscope. The vertical sweep frequencies are from about 60 cycles to 2,000 cycles, while the horizontal are from about

15,000 cycles to 500,000 cycles. The system illustrated in Fig. 44 is commonly used. The circuits illustrated in conjunction with the high-frequency output stage are known as *differentiating* circuits, whereas those associated with the low-frequency output stage are known as *integrating* circuits. In the former circuits, the voltage between the grid and the cathode of the tube is that across R_h (which may be an inductance if we wish). This voltage is *large* for high frequencies of a given amplitude. In the second circuit, the voltage between the grid and the cathode of the tube is that across C. This voltage is *large* for low frequencies of the *same* amplitude. Since, in a series circuit, Fig. 15, the voltage across L will be *larger* at high frequencies than at low frequencies, and the voltage across an inductance is given by the expression, $L\frac{di}{dt}$, a circuit which uses this feature is called a *differentiating circuit.* In the circuit of Fig. 45, the voltage across C will be *larger* at low frequencies than at high frequencies. Since the voltage across a condenser is given by the expression, $\frac{1}{C}\int idt$, a circuit which uses this feature is called an integrating circuit. The circuits of Fig. 44 are but two of the many television circuits whose design is directly traceable to the use of differential and integral calculus.

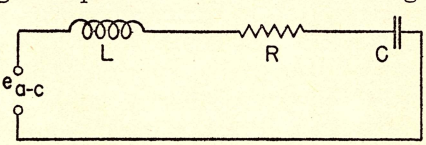


Fig. 45

Mathematical equations may readily be used to demonstrate the designations indicated for the circuits of Fig. 44. By Kirchhoff's law, it is evident that:

$$e = E_{Ch} + E_{Rh} = \frac{1}{C_h}\int idt + iR_h. \qquad \text{XXXVIa}$$

When $E_{Ch} \gg E_{Rh}$,

($\gg$ means "is much greater than".)

$$e \sim \frac{1}{C_h}\int idt;$$

($\sim$ means "approximately equals".)

by integration,

$$i = C_h\frac{de}{dt},$$

and by substitution,

$$E_{Rh} = iR_h = C_hR_h\frac{de}{dt}. \qquad \text{XXXVIb}$$

Since the voltage impressed on the grid of the high-frequency tube is a function of the derivative of the impressed voltage, the circuit is logically designated as a differentiating circuit.

By a similar analyais, we can show the low-frequency circuit of Fig. 44 to be an integrating circuit. Thus:

$$e=E_{C_l}+E_{R_l}=\frac{1}{C_l}\int i\,dt+iR_l \qquad \textbf{XXXVIc}$$

when $E_{R_l}\gg E_{C_l}$,

$$e\sim iR_l;$$

by algebra,

$$i\sim\frac{e}{R_l},$$

and by substitution,

$$E_{C_l}=\frac{1}{R_lC_l}\int e\,dt \qquad \textbf{XXXVId}$$

Since the voltage impressed on the grid of the low-frequency tube is a function of the integral of the impressed voltage, the circuit is logically designated as an integrating circuit.

SPECIAL CIRCUITS

In the earliest days of commercial radio, the early years of this century, the spark transmitter was the source of radio energy. The vacuum tube was a tremendous boon to radio because it could *amplify* radio signals in both transmitters and receivers. Of course, it could also be an oscillator, *i.e.*, a source of radio or audio frequencies, but this property was contingent upon its ability to amplify. Many circuits have since been developed in which the vacuum tube plays a major rôle, in which other properties are the primary considerations in the design of the circuits. We may consider a few of them now.

Voltage regulator

A vacuum tube may be used as a voltage regulator. Thus, the circuit of Fig. 46 is used in the auto alarm in radio rooms on ships to nullify the effect of variations in line voltage on the operating time-intervals of the distress-call receiver.

It is evident from Fig. 46 that the plate voltage E_p on the tube is:

$$E_p=E_{\text{line}}-e$$

and that the bias voltage, E_C is:

$$E_C=E_b-E_R.$$

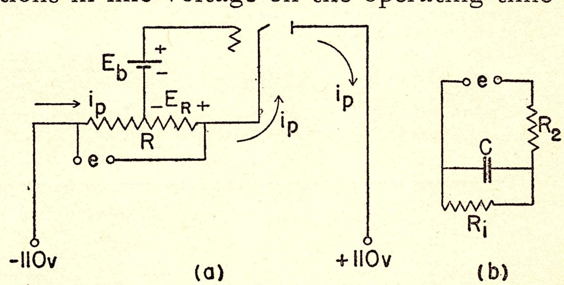


Fig. 46

Should the line voltage increase, the plate voltage will tend to increase, thus tending to increase the plate current, i_p, but increased current through the resistor, R, causes E_R to be greater in magnitude. This biases the tube more negatively, thus tending to decrease the plate current. A decrease in line voltage reverses the above action. The net result

is to keep the current through R rather constant and therefore the voltage, e, rather constant. If e is used to control other circuits, variations in line voltage will be of little importance. The voltage, e, is used to charge a condenser (one of several such circuits controlled), C, through resistor R_2. If the usual Kirchhoff's-law equations are written and solved as differential equations, the current is found to be:

$$i=\frac{e}{R_1+R_2}\left[1+\frac{R_1}{R_2}\varepsilon^{-\frac{(R_1+R_2)t}{(R_1R_2)C}}\right] \qquad \text{XXXVII}$$

By the selection of reasonable values for R_1, R_2, and C, the *time* required to build up the voltage across C to some designated value, may be fixed.

Automatic frequency control

The next circuit, shown in Fig. 47, may be the means of obtaining automatic frequency control in a receiver, or of modulating a frequency-modulation transmitter. The tube on the right may be the local oscillator in the receiver, or the signal source in the transmitter; the tube on the left looks like inductance in parallel with L_t and the amount of inductance it seems to be depends on the magnitude of *e control*. The mathematical statements to show this are as follows:

$$R \gg X_C$$

therefore, if

$$e_t = E_t \sin \omega t,$$

$$i_C = I_C \sin \omega t$$

because the impedance, $R-jX_C$, is practically pure R, so that the current through the branch, $R-C$, is practically in phase with the

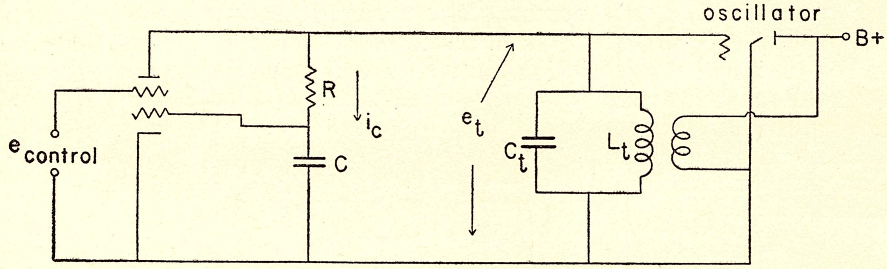


Fig. 47

voltage across it. However, the voltage across C is 90° *behind* the current and is therefore:

$$e_C = -E_C \cos \omega t.$$

Since the plate current that flows in the regulator tube is very nearly in phase with the control-grid voltage, the plate current lags the plate voltage by approximately 90°. The regulator tube, therefore, looks like inductance in parallel with the oscillator tank.

TEST YOUR ABILITY TO APPLY MATHEMATICS TO RADIO

1 Given the equation, $r \sin \theta + S \cos \theta = t$, show that $m \cos (\theta - \gamma) = t$ if $r = m \sin \gamma$ and $s = m \cos \gamma$. This work appears frequently in radio-circuit analyses.

2 The frequency of the carrier of a radio transmitter times the wave length of the carrier in meters is 3×10^8. If the frequency is 660 kc., what is the wave length?

3 To calculate the number of turns to wind on the secondary of an r-f transformer, we use this formula: $M = \frac{0.0501\, Nna^2}{\sqrt{l^2 + A^2}}$, where M = mutual inductance, N = number of turns on the primary winding, n = number of turns on the secondary winding, a = radius of the primary, A = radius of the secondary, l = length of the secondary. Solve the formula for n.

4 Prove that the power output in an A.-C. circuit with a resistive load is given by $0.5\, EI$.

5 Given three impedances in series, $10\underline{/30^\circ}$, $15\underline{/45^\circ}$, and $20\underline{/60^\circ}$. What is the equivalent impedance?

6 To calculate the inductance of an r-f solenoid, we may use the following formula: $L = \frac{r^2 n^2}{9r + 10l}$, where L = inductance in microhenries, r = radius of coil in inches, l = length of coil in inches = pn, n = number of turns, p = pitch, the space between turns. If the outside diameter of the form is $3\frac{3}{16}$ inches, the pitch is $\frac{3}{8}$ inch, and the inductance is 10 microhenries; how many turns are on the coil?

7 In a coaxial cable, the two conductors are concentric circles. How do any two chords of the greater circle, which are tangent to the smaller, compare in length?

8 Given a water-cooled vacuum tube with a tungsten filament 65 cm. long. When the temperature of the filament is 20° C., the resistance, R_o, is 0.283 ohms. If the resistance of a conductor is given by the formula, $R = R_o(1 + at)$, where t is the temperature change, what is the resistance of the filament at 1350° C.? ($a = 0.0051$ for tungsten.)

9 How much current flows in an A.-C. circuit where $E = 120$ volts, $R = 150$ ohms, $f = 1000$ cycles, $L = 0.03$ henries, and $C = 4 \times 10^{-6}$ farads?

$$I = \frac{E}{\sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}}$$

10 The number, 3×10^8, is commonly employed as the velocity of radio waves in meters per second. The number, 2.9982×10^8, is more accurate. What is the per cent of error introduced by using 3×10^8 in computations?

11 The determination of the insulation resistance of an insulated condutor by the leakage method uses the formula, $R = \frac{t}{C} \times \frac{10^6}{2.3 \log \frac{E_o}{E}}$. If $t = 108$ seconds, $E_o = 127$ volts, $E = 113$ volts, and $C = 0.087 \times 10^{-6}$ farads, what is R in ohms?

Solutions to Problems and Exercises in Issue 12

ELECTRICITY

ELECTRICAL UNITS

1 5,310 ft.-lb.
2 36 microamperes
3 4.3° C.
4 $19\frac{3}{4}$ minutes
5 8.04 watts
6 6.7 microhenrys
7 6.24×10^{12} electrons

RESISTANCE

8 (a) $I=0.2$ amperes
(b) 0.32 watts
9 2.4 amperes
10 0.2 sec.
11 $R=4.045$ ohms
12 $I=4.0$ amps
13 0.443 watts

VOLTAGE DROP AND RESISTIVITY

14 $E=375$ volts
15 $IR=1.88$ volts
16 $R=10$ ohms
17 0.218 ohms (Use Eq. XVII)
18 0.1285″
19 13,000 ohms

KIRCHHOFF'S LAWS

20 $I_1=\frac{4}{19}$ amperes
$I_2=\frac{4}{19}$ amperes
$I_3=\frac{8}{19}$ amperes

21 $I_1=90$ amps $E_1=101.9$ v.
$I_2=75$ amps $E_2=95.6$ v.
$I_3=15$ amps $E_3=107.9$ v.
$I_4=115$ amps $E_4=105.95$ v.
$I_5=140$ amps $E_5=183.55$ v.
$I_6=25$ amps—opposite direction to that in diagram

22 $I_1=60$ amps $E_1=117.28$ v.
$I_2=55$ amps $E_2=113.68$ v.
$I_3=5$ amps $E_3=116.32$ v.
$I_4=82$ amps $E_4=114.17$ v.
$I_5=78$ amps $E_5=215.85$ v.
$I_6=4$ amps

CAPACITY AND TIME CONSTANT

23 Disconnect 0.05 microfarad condensers
24 0.161 amps 0.011 amps
0.042 amps 0.003 amps
25 0.1728 henrys

IMPEDANCE

26 (a) 37.9 ohms. (b) capacitive reactance
27 $X=X_L-X_C=0.94-0.71=0.23$ ohms

FREQUENCIES

28 $f=159$ cycles per second
29 $X=44-19=25$ ohms, approx.
$Z=\sqrt{3600+625}=65$ ohms, approx.
$E_{30}=2\times65=130$ volts
$E_{45}=120.02$ volts
$E_{60}=124.2$ volts
$E_{90}=147$ volts

MILITARY GUNNERY

RANGE FINDERS

	Range	Error
1	2,000 yd.	± 9.6 yd.
	4,000 yd.	± 38.4 yd.
	6,000 yd.	± 86.4 yd.
	8,000 yd.	±153.6 yd.
	10,000 yd.	±240.0 yd.
2	5,000 yd.	± 20.0 yd.
	10,000 yd.	± 80.0 yd.
	15,000 yd.	±180.0 yd.
	20,000 yd.	±320.0 yd.

3 $\Delta\theta_A=1.6\times10^{-6}$ radians $=\frac{1''}{3}$

$\Delta\theta_B=2.4\times10^{-6}$ radians $=\frac{1''}{2}$

	Range	Error
4	4,000 yd.	± 12.8 yd.
	8,000 yd.	± 51.2 yd.
	12,000 yd.	±115.2 yd.
	16,000 yd.	±204.8 yd.
5	12,000 yd.	± 69.12 yd.
	16,000 yd.	±122.88 yd.
	24,000 yd.	±276.48 yd.
	32,000 yd.	±491.52 yd.

COMPUTING DIRECTION

6 350 yd.
7 Offset = 10, 20, 30 mils

8	Deg.	Mils
	10	177.78
	20	355.56
	30	533.33
	40	711.11
	50	888.89
	60	1,066.67
	70	1,244.44
	80	1,422.22
	90	1,600.00

9	Min.	Mils
	10	2.963
	20	5.926
	30	8.889
	40	11.852
	50	14.815
	60	17.778

FIRING ANGLES

10	Aiming Point	Piece	Firing Angle
	Front	left	$M+P-T$
	Front	right	$M+P-T$
	Rear	left	$M+P-T$
	Rear	right	$M+P-T$

11 $17\frac{1}{2}°$
12 132°
13 90°
} graphic solutions (approximate)

MILITARY GUNNERY (continued)

THE TRAJECTORY

14 $y=500t-16.1t^2$; $x=866t$
15 (a) $y=866t-16.1t^2$; $x=500t$
(b) $R=26{,}890'$; $R=26{,}890'$
(c) $h_{30°}=3{,}880'$; $h_{60°}=11{,}640'$
16 (a) 4 times original range
(b) 4 times original height
(c) 2 times original time

(*The remaining solutions, slide-rule accuracy.*)

17 (a) $y_{15°}=0.27x-\dfrac{x^2}{1.67\times10_5}$

$y_{30°}=0.58x-\dfrac{x^2}{1.34\times10_5}$

$y_{45°}=1.00x-\dfrac{x^2}{8.94\times10^4}$

$y_{60°}=1.73x-\dfrac{x^2}{4.47\times10_4}$

$y_{75°}=3.73x-\dfrac{x^2}{1.20\times10_4}$

	(c) *range*	(d) *elevation*
15°	44,700 ft.	3,000 ft.
30°	77,500 ft.	11,200 ft.
45°	89,400 ft.	22,400 ft.
60°	77,500 ft.	33,500 ft.
75°	44,700 ft.	41,700 ft.

(e) The angle of fall is the supplement of the firing angle:

165°, 150°, 135°, 120°, 105°

18 $y=0.58x-\dfrac{x^2}{14{,}910}$ $(v_0=100\sqrt{32})$

$y=0.58x-\dfrac{x^2}{59{,}640}$ $(v_0=200\sqrt{32})$

$y=0.58x-\dfrac{x^2}{134{,}190}$ $(v_0=300\sqrt{32})$

$y=0.58x-\dfrac{x^2}{238{,}560}$ $(v_0=400\sqrt{32})$

19 $r=2.21u$, $2.285u$

43	—	2.1%	$+4r$
147	—	7.2%	$+3r$
330	—	16.1%	$+2r$
497	—	24.3%	$+\ r$
			0
497	—	24.3%	$-\ r$
330	—	16.1%	$-2r$
147	—	7.2%	$-3r$
43	—	2.1%	$-4r$

Tables and Formulas

TABLE LXXVIII

SOLUTION OF SPHERICAL TRIANGLES

Problem	*Formula*
To find *hypotenuse*, given two legs	$\cos c = \cos a \cos b$
To find *hypotenuse*, given two angles	$\cos c = \cot A \cot B$
To find *leg*, given acute angle and leg opposite acute angle	$\begin{cases}\sin a = \tan b \cot B\\ \sin b = \tan a \cot A\end{cases}$
To find *leg*, given hypotenuse, acute angle	$\begin{cases}\sin b = \sin c \sin B\\ \sin a = \sin c \sin A\end{cases}$
To find *angle*, given leg and hypotenuse	$\begin{cases}\cos A = \tan b \cot c\\ \cos B = \tan a \cot c\end{cases}$
To find *angle*, given leg and adjacent acute angle	$\begin{cases}\cos B = \cos b \sin A\\ \cos A = \cos a \sin B\end{cases}$

TABLE LXXIX

GREENWICH A. M. 1941 JANUARY 1 (WEDNESDAY)

GCT	☉ SUN GHA	SUN Dec.	♈ GHA	♀ VENUS −3.4 GHA	VENUS Dec.	♃ JUPITER −2.2 GHA	JUPITER Dec.	♄ SATURN 0.4 GHA	SATURN Dec.	☽ MOON GHA	MOON Dec.
h m	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′
0 00	179 10	S23 03	100 15	207 34	S21 25	66 29	N12 19	63 51	N11 50	139 54	S11 37
10	181 40		102 46	210 04		68 59		66 21		142 19	36
20	184 10		105 16	212 34		71 30		68 51		144 44	35
30	186 40	· ·	107 46	215 03	· ·	74 00	· ·	71 22	· ·	147 09	· 33
40	189 09		110 17	217 33		76 30		73 52		149 34	32
50	191 39		112 47	220 03		79 01		76 23		151 59	31
1 00	194 09	S23 03	115 18	222 33	S21 25	81 31	N12 19	78 53	N11 50	154 24	S11 29
10	196 39		117 48	225 03		84 02		81 23		156 49	28
20	199 09		120 18	227 33		86 32		83 54		159 14	27
30	201 39	· ·	122 49	230 03	· ·	89 03	· ·	86 24	· ·	161 40	· 25
40	204 09		125 19	232 32		91 33		88 55		164 05	24
50	206 39		127 50	235 02		94 03		91 25		166 30	22
2 00	209 09	S23 03	130 20	237 32	S21 26	96 34	N12 19	93 56	N11 50	168 55	S11 21
10	211 39		132 51	240 02		99 04		96 26		171 20	20
20	214 09		135 21	242 32		101 35		98 56		173 45	18
30	216 39	· ·	137 51	245 02	· ·	104 05	· ·	101 27	· ·	176 10	· 17
40	219 09		140 22	247 32		106 35		103 57		178 36	16
50	221 39		142 52	250 01		109 06		106 28		181 01	14
3 00	224 09	S23 02	145 23	252 31	S21 26	111 36	N12 19	108 58	N11 50	183 26	S11 13
10	226 39		147 53	255 01		114 07		111 28		185 51	12
20	229 09		150 23	257 31		116 37		113 59		188 16	10
30	231 39	· ·	152 54	260 01	· ·	119 07	· ·	116 29	· ·	190 41	· 09
40	234 09		155 24	262 31		121 38		119 00		193 06	08
50	236 39		157 55	265 01		124 08		121 30		195 32	06
4 00	239 08	S23 02	160 25	267 30	S21 27	126 39	N12 19	124 01	N11 50	197 57	S11 05
10	241 38		162 55	270 00		129 09		126 31		200 22	03
20	244 08		165 26	272 30		131 39		129 01		202 47	02
30	246 38	· ·	167 56	275 00	· ·	134 10	· ·	131 32	· ·	205 12	11 01
40	249 08		170 27	277 30		136 40		134 02		207 37	10 59
50	251 38		172 57	280 00		139 11		136 33		210 03	58
5 00	254 08	S23 02	175 28	282 30	S21 27	141 41	N12 19	139 03	N11 50	212 28	S10 57
10	256 38		177 58	284 59		144 12		141 33		214 53	55
20	259 08		180 28	287 29		146 42		144 04		217 18	54
30	261 38	· ·	182 59	289 59	· ·	149 12	· ·	146 34	· ·	219 43	· 52
40	264 08		185 29	292 29		151 43		149 05		222 08	51
50	266 38		188 00	294 59		154 13		151 35		224 34	50
6 00	269 08	S23 02	190 30	297 29	S21 28	156 44	N12 19	154 06	N11 50	226 59	S10 48
10	271 38		193 00	299 59		159 14		156 36		229 24	47
20	274 08		195 31	302 28		161 44		159 06		231 49	46
30	276 38	· ·	198 01	304 58	· ·	164 15	· ·	161 37	· ·	234 14	· 44
40	279 08		200 32	307 28		166 45		164 07		236 39	43
50	281 38		203 02	309 58		169 16		166 38		239 05	41
7 00	284 08	S23 02	205 32	312 28	S21 28	171 46	N12 19	169 08	N11 50	241 30	S10 40
10	286 38		208 03	314 58		174 16		171 39		243 55	39
20	289 07		210 33	317 28		176 47		174 09		246 20	37
30	291 37	· ·	213 04	319 57	· ·	179 17	· ·	176 39	· ·	248 45	· 36
40	294 07		215 34	322 27		181 48		179 10		251 11	34
50	296 37		218 04	324 57		184 18		181 40		253 36	33
8 00	299 07	S23 01	220 35	327 27	S21 29	186 49	N12 19	184 11	N11 50	256 01	S10 32
10	301 37		223 05	329 57		189 19		186 41		258 26	30
20	304 07		225 36	332 27		191 49		189 11		260 51	29
30	306 37	· ·	228 06	334 57	· ·	194 20	· ·	191 42	· ·	263 17	· 27
40	309 07		230 37	337 26		196 50		194 12		265 42	26
50	311 37		233 07	339 56		199 21		196 43		268 07	25
9 00	314 07	S23 01	235 37	342 26	S21 29	201 51	N12 19	199 13	N11 50	270 32	S10 23
10	316 37		238 08	344 56		204 21		201 44		272 57	22
20	319 07		240 38	347 26		206 52		204 14		275 23	20
30	321 37	· ·	243 09	349 56	· ·	209 22	· ·	206 44	· ·	277 48	· 19
40	324 07		245 39	352 26		211 53		209 15		280 13	18
50	326 37		248 09	354 55		214 23		211 45		282 38	16
10 00	329 07	S23 01	250 40	357 25	S21 30	216 53	N12 19	214 16	N11 50	285 03	S10 15
10	331 37		253 10	359 55		219 24		216 46		287 29	13
20	334 07		255 41	2 25		221 54		219 16		289 54	12
30	336 37	· ·	258 11	4 55	· ·	224 25	· ·	221 47	· ·	292 19	· 11
40	339 06		260 41	7 25		226 55		224 17		294 44	09
50	341 36		263 12	9 55		229 26		226 48		297 09	08
11 00	344 06	S23 01	265 42	12 24	S21 30	231 56	N12 19	229 18	N11 50	299 35	S10 06
10	346 36		268 13	14 54		234 26		231 49		302 00	05
20	349 06		270 43	17 24		236 57		234 19		304 25	03
30	351 36	· ·	273 14	19 54	· ·	239 27	· ·	236 49	· ·	306 50	· 02
40	354 06		275 44	22 24		241 58		239 20		309 16	10 01
50	356 36		278 14	24 54		244 28		241 50		311 41	9 59
12 00	359 06	S23 01	280 45	27 24	S21 30	246 58	N12 19	244 21	N11 50	314 06	S9 58

☾'s Par.

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SD ☉ 16′

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Corr. HA ☾

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(From American Air Almanac)

TABLE LXXX

VI GREENWICH P. M. 1941 JANUARY 1 (WEDNESDAY)

GCT	☉ SUN GHA	☉ SUN Dec.	♈ GHA	♀ VENUS −3.4 GHA	♀ VENUS −3.4 Dec.	♃ JUPITER −2.2 GHA	♃ JUPITER −2.2 Dec.	♄ SATURN 0.4 GHA	♄ SATURN 0.4 Dec.	☽ MOON GHA	☽ MOON Dec.
h m	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′
12 00	359 06	S23 01	280 45	27 24	S21 30	246 58	N12 19	244 21	N11 50	314 06	S9 58
10	1 36		283 15	29 54		249 29		246 51		316 31	56
20	4 06		285 46	32 23		251 59		249 21		318 57	55
30	6 36	. .	288 16	34 53	. .	254 30	. .	251 52	. .	321 22	. 53
40	9 06		290 46	37 23		257 00		254 22		323 47	52
50	11 36		293 17	39 53		259 30		256 53		326 12	51
13 00	14 06	S23 00	295 47	42 23	S21 31	262 01	N12 19	259 23	N11 50	328 38	S9 49
10	16 36		298 18	44 53		264 31		261 54		331 03	48
20	19 06		300 48	47 23		267 02		264 24		333 28	46
30	21 36	. .	303 18	49 52	. .	269 32	. .	266 54	. .	335 53	. 45
40	24 06		305 49	52 22		272 02		269 25		338 19	43
50	26 36		308 19	54 52		274 33		271 55		340 44	42
14 00	29 06	S23 00	310 50	57 22	S21 31	277 03	N12 19	274 26	N11 50	343 09	S9 41
10	31 35		313 20	59 52		279 34		276 56		345 34	39
20	34 05		315 51	62 22		282 04		279 26		348 00	38
30	36 35	. .	318 21	64 52	. .	284 35	. .	281 57	. .	350 25	. 36
40	39 05		320 51	67 21		287 05		284 27		352 50	35
50	41 35		323 22	69 51		289 35		286 58		355 15	33
15 00	44 05	S23 00	325 52	72 21	S21 32	292 06	N12 19	289 28	N11 50	357 41	S9 32
10	46 35		328 23	74 51		294 36		291 59		0 06	31
20	49 05		330 53	77 21		297 07		294 29		2 31	29
30	51 35	. .	333 23	79 51	. .	299 37	. .	296 59	. .	4 57	. 28
40	54 05		335 54	82 21		302 07		299 30		7 22	26
50	56 35		338 24	84 50		304 38		302 00		9 47	25
16 00	59 05	S23 00	340 55	87 20	S21 32	307 08	N12 19	304 31	N11 50	12 12	S9 23
10	61 35		343 25	89 50		309 39		307 01		14 38	22
20	64 05		345 55	92 20		312 09		309 32		17 03	20
30	66 35	. .	348 26	94 50	. .	314 39	. .	312 02	. .	19 28	. 19
40	69 05		350 56	97 20		317 10		314 32		21 53	18
50	71 35		353 27	99 50		319 40		317 03		24 19	16
17 00	74 05	S23 00	355 57	102 19	S21 33	322 11	N12 19	319 33	N11 50	26 44	S9 15
10	76 35		358 28	104 49		324 41		322 04		29 09	13
20	79 05		0 58	107 19		327 12		324 34		31 35	12
30	81 34	. .	3 28	109 49	. .	329 42	. .	327 04	. .	34 00	. 10
40	84 04		5 59	112 19		332 12		329 35		36 25	09
50	86 34		8 29	114 49		334 43		332 05		38 51	07
18 00	89 04	S22 59	11 00	117 19	S21 33	337 13	N12 19	334 36	N11 50	41 16	S9 06
10	91 34		13 30	119 48		339 44		337 06		43 41	04
20	94 04		16 00	122 18		342 14		339 37		46 06	03
30	96 34	. .	18 31	124 48	. .	344 44	. .	342 07	. .	48 32	. 02
40	99 04		21 01	127 18		347 15		344 37		50 57	9 00
50	101 34		23 32	129 48		349 45		347 08		53 22	8 59
19 00	104 04	S22 59	26 02	132 18	S21 34	352 16	N12 19	349 38	N11 50	55 48	S8 57
10	106 34		28 32	134 48		354 46		352 09		58 13	56
20	109 04		31 03	137 17		357 16		354 39		60 38	54
30	111 34	. .	33 33	139 47	. .	359 47	. .	357 09	. .	63 04	. 53
40	114 04		36 04	142 17		2 17		359 40		65 29	51
50	116 34		38 34	144 47		4 48		2 10		67 54	50
20 00	119 04	S22 59	41 05	147 17	S21 34	7 18	N12 19	4 41	N11 50	70 20	S8 48
10	121 34		43 35	149 47		9 49		7 11		72 45	47
20	124 04		46 05	152 17		12 19		9 42		75 10	45
30	126 34	. .	48 36	154 46	. .	14 49	. .	12 12	. .	77 35	. 44
40	129 04		51 06	157 16		17 20		14 42		80 01	42
50	131 33		53 37	159 46		19 50		17 13		82 26	41
21 00	134 03	S22 59	56 07	162 16	S21 35	22 21	N12 19	19 43	N11 50	84 51	S8 40
10	136 33		58 37	164 46		24 51		22 14		87 17	
20	139 03		61 08	167 16		27 21		24 44		89 42	
30	141 33	.	63 38	169 46	. .	29 52	. .	27 14	. .	92 07	. 35
40	144 03		66 09	172 15		32 22		29 45		94 33	34
50	146 33		68 39	174 45		34 53		32 15		96 58	32
22 00	149 03	S22 59	71 09	177 15	S21 35	37 23	N12 19	34 46	N11 50	99 23	S8 31
10	151 33		73 40	179 45		39 53		37 16		101 49	29
20	154 03		76 10	182 15		42 24		39 47		104 14	28
30	156 33	. .	78 41	184 45	. .	44 54	. .	42 17	. .	106 39	. 26
40	159 03		81 11	187 15		47 25		44 47		109 05	25
50	161 33		83 41	189 44		49 55		47 18		111 30	23
23 00	164 03	S22 58	86 12	192 14	S21 35	52 26	N12 19	49 48	N11 50	113 55	S8 22
10	166 33		88 42	194 44		54 56		52 19		116 21	20
20	169 03		91 13	197 14		57 26		54 49		118 46	19
30	171 33	. .	93 43	199 44	. .	59 57	. .	57 19	. .	121 11	. 17
40	174 03		96 14	202 14		62 27		59 50		123 37	16
50	176 33		98 44	204 44		64 58		62 20		126 02	14
24 00	179 03	S22 58	101 14	207 13	S21 36	67 28	N12 19	64 51	N11 50	128 28	S8 13

Lat.	Sunrise	Twlt.	Moonrise	Diff.
° N	h m	m	h m	m
60	9 03	57	10 14	19
58	8 46	51	08	21
56	32	47	10 03	23
54	19	43	9 58	25
52	8 08	40	54	26
50	7 59	38	50	28
45	38	33	42	30
40	22	31	35	33
35	7 08	28	29	35
30	6 56	27	23	38
20	35	24	14	41
10	17	23	9 06	43
0	6 00	22	8 58	46
10	5 43	23	50	50
20	24	24	42	52
30	5 03	27	33	55
35	4 50	29	28	57
40	35	33	21	60
45	4 18	36	14	62
50	3 56	43	06	65
52	45	47	8 02	66
54	33	52	7 57	68
56	19	60	52	70
58	3 03	71	47	72
60	2 43	93	7 41	74
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Lat.	Sunset	Twlt.	Moonset	Diff.
° N	h m	m	h m	m
60	15 05	57	20 16	74
58	21	51	22	71
56	36	47	26	70
54	48	43	30	68
52	15 59	40	34	66
50	16 08	37	37	65
45	29	33	44	62
40	45	31	50	59
35	16 59	28	20 56	56
30	17 11	27	21 00	54
20	32	24	08	51
10	17 50	22	14	48
0	18 07	22	21	45
10	18 25	23	27	43
20	18 43	25	34	39
30	19 05	27	41	36
35	18	29	46	34
40	32	33	51	31
45	19 50	37	21 56	30
50	20 12	44	22 03	26
52	22	49	06	25
54	34	54	09	24
56	20 48	62	13	22
58	21 04	73	17	20
60	21 24	96	22 22	18
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(From American Air Almanac)

TABLE LXXXI

STARS

Alphabetical order					Order of SHA		
Name	Mag.	SHA	Dec.	RA	SHA	Dec.	Name
		° ′	° ′	h m	° ′	° ′	
Acamar	3.4	316 00	S40 33	2 56	14 33	N14 53	Markab
Achernar	0.6	336 08	S57 32	1 35	16 24	S29 56	Fomalhaut
Acrux	1.6	174 10	S62 46	12 23	28 52	S47 15	Al Na'ir
Adhara	1.6	255 55	S28 54	6 56	34 41	N 9 36	Enif
Aldebaran . (a) . .	1.1	291 52	N16 23	4 33	50 09	N45 04	Deneb
Alioth	1.7	167 08	N56 17	12 51	54 45	S56 55	Peacock
Al Na'ir	2.2	28 52	S47 15	22 5	63 01	N 8 43	Altair
Alnilam	1.8	276 42	S 1 15	5 33	77 06	S26 22	Nunki
Alphard	2.2	218 49	S 8 24	9 25	81 16	N38 44	Vega
Alphecca	2.3	126 57	N26 55	15 32	84 56	S34 25	Kaus Aust.
Alpheratz	2.2	358 40	N28 46	0 5	91 12	N51 30	Etamin
Altair	0.9	63 01	N 8 43	19 48	96 57	N12 36	Rasalague
Al Suhail	2.2	223 32	S43 12	9 6	97 36	S37 04	Shaula
Antares . (d) . .	1.2	113 33	S26 18	16 26	103 15	S15 39	Sabik
Arcturus	0.2	146 45	N19 29	14 13	(109 24)	S68 55	α Tri. Aust.
ε Argus	1.7	234 40	S59 19	8 21	113 33	S26 18	Antares
Bellatrix	1.7	279 30	N 6 18	5 22	120 47	S22 27	Dschubba
Betelgeux	0.1–1.2	272 00	N 7 24	5 52	126 57	N26 55	Alphecca
Canopus	−0.9	264 20	S52 40	6 23	(137 17)	N74 24	Kochab
Capella	0.2	281 55	N45 56	5 12	141 06	S60 35	Rigil Kent.
Caph	2.4	358 30	N58 50	0 6	146 45	N19 29	Arcturus
θ Centauri	2.3	149 12	S36 05	14 3	149 12	S36 05	θ Centauri
β Crucis	1.5	168 55	S59 22	12 44	159 28	S10 51	Spica
γ Crucis	1.6	173 01	S56 47	12 28	159 36	N55 14	Mizar
Deneb	1.3	50 09	N45 04	20 39	167 08	N56 17	Alioth
Denebola	2.2	183 29	N14 54	11 46	168 55	S59 22	β Crucis
Deneb Kait.	2.2	349 51	S18 19	0 41	173 01	S56 47	γ Crucis
Dubhe	2.0	194 58	N62 04	11 0	174 10	S62 46	Acrux
Dschubba	2.5	120 47	S22 27	15 57	183 29	N14 54	Denebola
Enif	2.5	34 41	N 9 36	21 41	194 58	N62 04	Dubhe
Etamin	2.4	91 12	N51 30	17 55	208 41	N12 15	Regulus
Fomalhaut	1.3	16 24	S29 56	22 54	218 49	S 8 24	Alphard
Hamal	2.2	329 02	N23 11	2 4	221 51	S69 29	Miaplacidus
Kaus Aust..	2.0	84 56	S34 25	18 20	223 32	S43 12	Al Suhail
Kochab	2.2	(137 17)	N74 24	14 51	234 40	S59 19	ε Argus
Marfak	1.9	309 58	N49 39	3 20	244 34	N28 10	Pollux
Markab	2.6	14 33	N14 53	23 2	245 56	N 5 22	Procyon
Miaplacidus	1.8	221 51	S69 29	9 13	255 55	S28 54	Adhara
Mizar	2.4	159 36	N55 14	13 22	259 22	S16 38	Sirius
Nunki	2.1	77 06	S26 22	18 52	264 20	S52 40	Canopus
Peacock	2.1	54 45	S56 55	20 21	272 00	N 7 24	Betelgeux
Polaris	2.1	(334 12)	N88 59	1 43	276 42	S 1 15	Alnilam
Pollux	1.2	244 34	N28 10	7 42	279 30	N 6 18	Bellatrix
Procyon	0.5	245 56	N 5 22	7 36	281 55	N45 56	Capella
Rasalague	2.1	96 57	N12 36	17 32	282 04	S 8 16	Rigel
Regulus . . (b) . .	1.3	208 41	N12 15	10 5	291 52	N16 23	Aldebaran
Rigel	0.3	282 04	S 8 16	5 12	309 58	N49 39	Marfak
Rigil Kent..	0.3	141 06	S60 35	14 36	316 00	S40 33	Acamar
Ruchbah	2.8	339 31	N59 56	1 22	329 02	N23 11	Hamal
Sabik	2.6	103 15	S15 39	17 7	(334 12)	N88 59	Polaris
Shaula	1.7	97 36	S37 04	17 30	336 08	S57 32	Achernar
Sirius	−1.6	259 22	S16 38	6 43	339 31	N59 56	Ruchbah
Spica . . . (c) . .	1.2	159 28	S10 51	13 22	349 51	S18 19	Deneb Kait.
α Tri. Aust.	1.9	(109 24)	S68 55	16 42	358 30	N58 50	Caph
Vega	0.1	81 16	N38 44	18 35	358 40	N28 46	Alpheratz

SHA = 360° − RA

Jan.-Apr., 1941

(From American Air Almanac)

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